Partial Differential Equation

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MTM-402: Classification, Characteristic equation, Some important linear Partial Differential Equations, Fundamental solution of Laplace equation, Mean value theorem, Harmonic functions and properties, Representation formula, Green's functions, Green representation formula, Poisson representation formula, Solution of Dirichlet's problem on the ball, Sub-Harmonic functions, The maximum principle, Energy methods, Fundamental solutions of Heat equations, Mean value formula, Properties of solutions, Initial value problem, B.V.Problem for Heat equation, Wave equation, Mean Value method, Solution of Wave equation with initial values, energy methods.

(i) **Heat equation:** (Parabolic equation)

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

(ii) **Laplace equation:** (Elliptic equation)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

(iii) **Wave equation:** (Hyperbolic equation)

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

We have,

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$$

is general equation of second order linear P.D.E and it will be
(i) Parabolic if $B^2 - 4AC = 0$
(ii) Hyperbolic if $B^2 - 4AC > 0$
(iii) Elliptic if $B^2 - 4AC < 0$.

Consider second order linear partial differential equation

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G. \qquad (1)$$

Let ξ and η are two different variable or varient(dependent variable) i.e $\xi = \xi(x, y)$ & $\eta(x, y)$.
We transform (x, y) to (ξ, η) i.e. $(x, y) \longmapsto (\xi, \eta)$, provided Jacobian of $(\xi, \eta) \neq 0$

$$\text{i.e., } J = \begin{bmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{bmatrix} \neq 0,$$

i.e., J must be invertible so that we may also go back from (ξ,η) plane to (x,y) plane.

$$\begin{aligned}
\text{Now, } u_x &= \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} \\
u_x &= u_\xi \xi_x + u_\eta \eta_x \qquad (2) \\
u_y &= \frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} \\
u_y &= u_\xi \xi_y + u_\eta \eta_y \qquad (3) \\
u_{xy} &= (u_x)_y = \frac{\partial}{\partial y}(u_x) = \frac{\partial}{\partial y}(u_\xi \xi_x + u_\eta \eta_x) \\
&= \frac{\partial}{\partial y}(u_\xi) + u_\xi \frac{\partial}{\partial y}(\xi_x) + \frac{\partial}{\partial y}(u_\eta)\eta_x + u_\eta \frac{\partial}{\partial y}(\eta_x) \\
&= \left[\frac{\partial}{\partial \xi}(u_\xi)\frac{\partial \xi}{\partial y} + \frac{\partial}{\partial \eta}(u_\xi)\frac{\partial \eta}{\partial y}\right]\xi_x + u_\xi \frac{\partial^2 \xi}{\partial y \partial x} + \left[\frac{\partial}{\partial \xi}(u_\eta)\frac{\partial \xi}{\partial y} + \frac{\partial}{\partial \eta}(u_\eta)\frac{\partial \eta}{\partial y}\right]\eta_x + u_\eta(\eta_{xy}) \\
u_{xy} &= u_{\xi\xi}\xi_x\xi_y + u_{\xi\eta}(\eta_y\xi_x + \xi_y\eta_x) + u_{\eta\eta}\eta_x\eta_y + u_\xi \xi_{xy} + u_\eta \eta_{xy}. \\
\text{Now, } u_{xx} &= (u_x)_x = (u_\xi \xi_x + u_\eta \eta_x)_x = (u_\xi \xi_x)_x + (u_\eta \eta_x)_x \\
(u_\xi)_x &= \frac{\partial}{\partial \xi}(u_\xi)\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}(u_\xi)\frac{\partial \eta}{\partial x} = u_{\xi\xi}\xi_x + u_{\eta\xi}\eta_x \\
&= u_\xi \xi_{xx} + \xi_x(u_{\xi\xi}\xi_x + u_{\eta\xi}\eta_x) + u_\eta \eta_{xx} + \eta_x(u_\eta)_x \\
\text{and } (u_\eta)_x &= \frac{\partial}{\partial x}\frac{\partial u}{\partial \eta} = \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \eta}\right)\frac{\partial \eta}{\partial x} + \frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \eta}\right)\frac{\partial \xi}{\partial x} = u_{\eta\eta}\eta_x + u_{\xi\eta}\xi_x \\
u_{xx} &= u_\xi \xi_{xx} + \xi_x(u_{\xi\xi}\xi_x + u_{\eta\xi}\eta_x) + u_\eta \eta_{xx} + \eta_x(u_{\eta\eta}\eta_x + u_{\xi\eta}\xi_x)
\end{aligned}$$

Solution of 1^{st} order linear homogeneous equations in $\mathbb{R}^2$:

Consider a simple homogeneous equation with constant coefficients

$$au_x + bu_y = 0 \quad \Leftrightarrow \quad (a,b).Du = 0. \qquad (1)$$

The linear equation says that the directional derivative of u in the direction of $V = (a,b)$ is zero. This means that the function $u(x,y)$ remains constant on lines in the direction of (a,b).
The equations of such lines are

$$\begin{aligned}
(x - x_0, y - y_0).(b,a) &= 0 \\
bx - ay - bx_0 + ay_0 &= 0 \\
\text{or } \{bx - ay = c | c \in \mathbb{R}\}
\end{aligned}$$

and are called characteristic lines for (1). If we think of a line as a function of x, then

$$\frac{dy}{dx} = \frac{b}{a} \ (a \neq 0).$$

Now if u does not change along those lines, then

$$\begin{aligned}
u(x,y)\,|_{bx-ay=c} &= f(c) \\
\Rightarrow u(x,y) &= f(bx - ay)
\end{aligned}$$

If one wants a more precise description of f then some conditions must be specified. Now let us consider $n-$ dimensional transport equation with constant coefficient

$$u_t + b.Du = 0 \quad \text{in} \ \mathbb{R}^n \times (0,\infty) \qquad (1)$$

where b is a fixed vector in $\mathbb{R}^n$, $b = (b_1, b_2, ..., b_n)$, and $u : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ is the unknown, $u = u(x,t)$.
Here $x = (x_1, x_2, ...x_n) \in \mathbb{R}^n$ denotes a typical point in space and $t \geq 0$ denotes a typical time. We write $Du = D_x u = (u_{x_1}, u_{x_2}, ..., u_{x_n})$ for the gradient of u with respect to the spatial variable x.
The partial differential equation (1) asserts that a particular directional derivatives of u vanishes. We use this fact by fixing any point $(x,t) \in \mathbb{R}^n \times [0,\infty)$ and defining

$$z(s) = u(x+sb, t+s) \quad (s \in \mathbb{R})$$

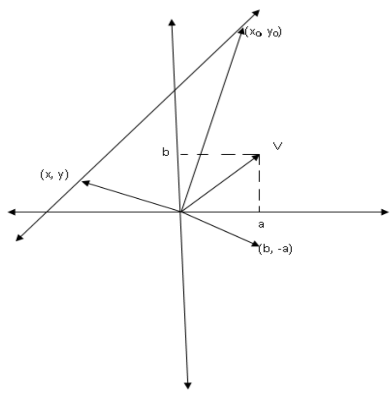
We then calculate

$$\frac{dz}{ds} = Du(x+sb, t+s).b + u_t(x+sb, t+s) = 0.$$

The second equality holds because of (1) holds. Thus $z(s)$ is a constant function of s and consequently for each point (x,t), u is constant on the line through (x,t) with the direction $(b,1) \in \mathbb{R}^{n+1}$. Hence if we know the value of u at any point on each such line, we know the value every where in $\mathbb{R}^n \times [0,\infty)$.

<u>Initial Value Problem:</u> To be precise, let us consider the following initial value problem

$$\begin{aligned} u_t + b.Du &= 0 \ \text{ in } \ \mathbb{R}^n \times (0,\infty) \qquad (2) \\ u &= g \ \text{ on } \ \mathbb{R}^n \times \{t=0\}. \end{aligned}$$



Here $b \in \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}$ are known, and the problem is to compute u. Given (x,t) as above, the line through (x,t) with direction $(b,1)$ is represented parametrically by $(x+sb, t+s)$ $(s \in \mathbb{R})$. This lines hits the plane $\Gamma = \mathbb{R}^n \times \{t=0\}$ when $s+t=0$, i.e. $s=-t$, at the point $(x-tb, 0)$. Since u is constant on the line and $u(x-bt,0) = g(x-tb)$, we deduce

$$u(x,t) = g(x-tb) \quad (x \in \mathbb{R}^n, t \geq 0). \qquad (3)$$

So if (2) has a sufficient regular solution u, it must certainly be given by (3). And conversely, it is easy to check directly that if $g \in C^1$, then u defined by (3) is indeed a solution of (2)

<u>Remark:</u> If g is not C^1, then there is obviously no C^1 solution of (2). But even in this case formula (3) certainly provides a stronger and in fact the only reasonable candidate for the solution. We may thus informly declare that $u(x,t) = g(x-tb), (x \in \mathbb{R}^n, t \geq 0)$

to be a weak solution of (2), even should g not be C^1.

Nonhomogeneous Problem:

Let us now look at the associated nonhomogeneous problem

$$\begin{aligned} u_t + b.Du &= f \quad \text{in } \mathbb{R}^n \times (0,\infty) \\ u &= g \quad \text{on } \mathbb{R}^n \times \{t=0\}. \end{aligned} \tag{4}$$

As before fix $(x,t) \in \mathbb{R}^{n+1}$, and inspired by the calculation above. Set

$$\begin{aligned} z(s) &:= u(x+sb, t+s) \quad \text{for } s \in \mathbb{R}. \text{ Then} \\ \dot{\mathcal{Z}}(s) &= Du(x+sb,t+s).b + u_t(x+sb,t+s) = f(x+sb,t+s) \end{aligned}$$

Consequently

$$\begin{aligned} u(x,t) - g(x-tb) &= u(x,t) - u(x,0) \\ &= z(0) - z(-t) \\ &= \int_{-t}^{0} \dot{\mathcal{Z}}(s) ds \\ &= \int_{-t}^{0} f(x+sb,t+s) ds \\ &= \int_{0}^{t} f(x+(s-t)b,s) ds \end{aligned}$$

and so

$$u(x,t) = g(x-tb) + \int_0^t f(x+(s-t)b,s)ds, \quad (x \in \mathbb{R}^n, t \geq 0), \tag{5}$$

solves the initial value problem (4).

Surface and Volume of a Hyper-Sphere:

Let $|A| \neq 0$ and consider the multiple integral

$$I = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} e^{-\langle Ax,Ax\rangle} dx_1 dx_2 ... dx_n.$$

If we put $y = Ax$, then $x = A^{-1}y$, and so

$$\begin{aligned} I &= \int_{\mathbb{R}^n} e^{-\langle Ax,Ax\rangle} dV(x) = \int_{\mathbb{R}^n} e^{-\langle y,y\rangle} |det\left(\frac{\partial x}{\partial y}\right)| dV(y) \\ &= \int_{\mathbb{R}^n} e^{-\langle y,y\rangle} |det(A^{-1})| dy_1 dy_2 ... dy_n \\ &= |det(A^{-1})| \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} e^{-(y_1^2+y_2^2+...+y_n^2)} dy_1 dy_2 ... dy_n \\ &= \frac{1}{|det(A^{-1})}| \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} e^{-(y_1^2+y_2^2+...+y_n^2)} dy_1 dy_2 ... dy_n \\ &= \frac{1}{|det(A^{-1})}| \int_{-\infty}^{\infty} e^{-y_1^2} dy_1 \int_{-\infty}^{\infty} e^{-y_2^2} dy_2 ... \int_{-\infty}^{\infty} e^{-y_n^2} dy_n \end{aligned}$$

But as

$$J=\int_{-\infty}^{\infty} e^{-t^2}dt = 2\int_{0}^{\infty} e^{-t^2}dt = \int_{0}^{\infty} e^{-s}s^{\frac{1}{2}-1}ds = \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

it follows that

$$I=\frac{\pi^{\frac{n}{2}}}{|det(A)|}=\frac{\pi^{\frac{n}{2}}}{|\lambda_1(A)\lambda_2(A)...\lambda_n(A)|}$$

Corollary 1: Let B be a positive definite matrix. Then

$$\int_{\mathbb{R}^n} e^{-\langle Bx,x\rangle}dV(x)=\frac{\pi^{\frac{n}{2}}}{|\lambda_1(B)\lambda_2(B)...\lambda_n(B)|^{\frac{1}{2}}}=\frac{\pi^{\frac{n}{2}}}{|det(B)|^{\frac{1}{2}}}$$

Proof: Let $A=\sqrt{B}$. Then $\langle Bx,x\rangle=\langle Ax,Ax\rangle$ and $\lambda_i(A)=\sqrt{\lambda_i(B)}$ for $i=1(1)n$. i.e B must be positive definite as $\langle Bx,x\rangle=\langle A^2x,x\rangle=\langle Ax,Ax\rangle$.

Corollary 2: Let $A_n(\rho)$ denotes the surface area of the n-dimensional sphere $S_n(\rho)$ of radius ρ. Then

$$A_n(\rho)=\frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}\rho^{n-1}$$

Proof: Taking $B=I$, the identity matrix

$$\begin{aligned}\pi^{\frac{n}{2}}=\int_{\mathbb{R}^n} e^{-\langle x,x\rangle}dV &= \int_0^\infty\int_{A_n(r)} e^{-\langle x,x\rangle}dA_n(r)dr\\ &= \int_0^\infty\int_{A_n(1)} e^{-r^2}r^{n-1}dA_n(1)dr\end{aligned}$$

$$\begin{aligned}\pi^{\frac{n}{2}} &= A_n(1)\int_0^\infty e^{-r^2}r^{n-1}dr\\ &= A_n(1)\int_0^\infty e^{-s}s^{\frac{n}{2}-1}(\frac{1}{2})ds=\frac{\Gamma(\frac{n}{2})}{2}A_n(1)\end{aligned}$$

Thus $A_n(1)=\frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$. Since $A_n(\rho)=\rho^{n-1}A_n(1)$. So

$$A_n(\rho)=\frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}\rho^{n-1}.$$

Remark: Note that $A_2(r)=2\pi r$, the circumference of a circle of radius r, $A_3(r)=\frac{2\pi^{\frac{3}{2}}r^2}{\frac{1}{2}\sqrt{\pi}}=4\pi r^2$, the area of the surface of sphere of radius r, which are the familiar formulae. Note that $A_1(r)=2$, the number of the end point of the line segment $[-r,r]$.

Corollary3: The volume $V_n(\rho)$ of the $n-$ dimensional hypersphere of radius ρ is given by

$$V_n(\rho)=\frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}\rho^n$$

Proof:

$$\begin{aligned} V_n(\rho) &= \int_0^\rho A_n(r)dr = \int_0^\rho \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} r^{n-1}dr \\ &= \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}\frac{\rho^n}{n} = \frac{\pi^{\frac{n}{2}}}{\frac{n}{2}\Gamma(\frac{n}{2})}\rho^n = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}\rho^n. \end{aligned}$$

Remark: $V_1(r) = \frac{\sqrt{\pi}}{\frac{1}{2}\sqrt{\pi}}r = 2r$, the volume of $1-$ sphere of radius r i.e. the length of the interval $\{x : |x| \leq r\}$, $V_2(r) = \frac{\pi}{1}r^2 = \pi r^2$, area inside a circle of radius r, and

$$V_3(r) = \frac{\pi^{\frac{3}{2}}}{\Gamma(\frac{3}{2})+1}r^3 = \frac{\pi^{\frac{3}{2}}}{\frac{3}{2}\frac{1}{2}\sqrt{\pi}}r^3 = \frac{4}{3}\pi r^3,$$

the familiar formula for the volume of a sphere of radius r.

Laplace equation: In two dimension

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

In n-dimensional space

$$\begin{aligned} \frac{\partial^2 u}{\partial {x_1}^2} + \frac{\partial^2 u}{\partial {x_2}^2} + \frac{\partial^2 u}{\partial {x_3}^2} + \ldots + \frac{\partial^2 u}{\partial {x_n}^2} &= 0 \\ \text{or } \Delta u = 0 \qquad\qquad (1) \end{aligned}$$

Here, unknown $u : U \longrightarrow \mathbb{R}$, $u = u(x)$ where $U \subseteq \mathbb{R}^n$ is given open set and Poisson's equation is $-\Delta u = f \qquad (2)$

Physical interpretation: Laplace equation comes up in a wide variety of physical context. u denotes the density of some quantity (e.g. a chemical concentration) in equilibrium. Then V is any smooth subregion within U, U be any open set in $\mathbb{R}^n$ then the net flux through ∂V is zero.

$$\int_{\partial V} \overrightarrow{F}.\nu dS = 0.$$

Here F denoting the flux density and ν the unit outward normal field. By Gauss Green theorem

$$\int_V div\overrightarrow{F}dx = \int_{\partial V} \overrightarrow{F}.\nu dS = 0$$

since V is arbitrary and so

$$div\overrightarrow{F} = 0.$$

In many physical situations it is found that $\overrightarrow{F}$ is proportional to gradient of u (i.e. Du) but points in opposite direction(since the flow is from region of higher to lower concentration). Thus

$$
\begin{aligned}
\overrightarrow{F} &= -\alpha Du \quad (\alpha > 0) \qquad\qquad (4)\\
div(Du) &= \overrightarrow{D} \cdot (Du) = \triangle u = 0 \quad [\text{ Laplace equation}].
\end{aligned}
$$

$$
\text{If } u \text{ denotes} \quad \begin{cases} \text{Chemical concentration} \\ \\ \text{temperature} \\ \\ \text{electrostatic potential} \end{cases}
$$

$$
\text{equation (4) is} \quad \begin{cases} \text{Fick's law of diffusion} \\ \\ \text{Fourier law of heat conduction} \\ \\ \text{Ohm's law of electrical conduction} \end{cases}
$$

Fundamental solution:

(a) Derivation of fundamental solution:

Since Laplace equation is invariant under rotation. Let $x = (x_1, x_2, x_3, \ldots, x_n) \in \mathbb{R}^n \ U \subseteq \mathbb{R}^n$. Now for $\Delta u = 0 \quad (1)$

$$
\Rightarrow \sum_{i=1}^{n} u_{x_i x_i} = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = 0
$$

We have to find radial solution. Here

$$
\begin{aligned}
r &= |x| = \sqrt{x_1^2 + x_2^2 + x_3^2 + \ldots + x_n^2}\\
r^2 &= x_1^2 + x_2^2 + x_3^2 + \ldots + x_n^2.
\end{aligned}
$$

Let us therefore attempt to find a solution u of Laplace equation in $U \subseteq \mathbb{R}^n$ having the form $u(x) = v(r)$

$$
\begin{aligned}
\text{where } r &= |x| = \sqrt{x_1^2 + x_2^2 + x_3^2 + \ldots + x_n^2}\\
\frac{\partial r}{\partial x_i} &= \frac{\partial x_i}{\partial r} = \frac{x_i}{r}, \quad x_i \neq 0\\
u_{x_i} &= \frac{\partial u}{\partial x_i} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial x_i} = v'(r)\left(\frac{x_i}{r}\right)\\
u_{x_i} &= v'(r)\left(\frac{x_i}{r}\right)\\
u_{x_i x_i} &= \frac{\partial}{\partial x_i}(u_{x_i}) = \frac{\partial}{\partial x_i}\left(v'(r)\frac{x_i}{r}\right)\\
&= \frac{\partial}{\partial x_i}\left(v'(r)\right)\frac{x_i}{r} + v'(r)\frac{\partial}{\partial x_i}\left(\frac{x_i}{r}\right)\\
&= \frac{\partial}{\partial r}v'(r)\frac{\partial r}{\partial x_i}\left(\frac{x_i}{r}\right) v'(r)\left(\frac{1}{r} + x_i\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\frac{\partial r}{\partial x_i}\right)\\
&= v''(r)\left(\frac{x_i}{r}\right)^2 + v'(r)\left(\frac{1}{r} - \frac{x_i}{r^2}\left(\frac{x_i}{r}\right)\right).
\end{aligned}
$$

Now,

$$
\begin{aligned}
\sum u_{x_i x_i} &= \sum v''(r)\left(\frac{x_i}{r}\right)^2 + \sum v'(r)\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right) \\
\Delta u &= v''(r)\frac{r^2}{r^2} + v'(r)\frac{n}{r} - v'(r)\frac{r^2}{r^3} \\
0 &= v''(r) + \frac{n-1}{r}v'(r) \\
\text{or} \qquad & v''(r) + \frac{n-1}{r}v'(r) = 0 \\
\text{If, } \ v' \neq 0 \qquad & (\log v')' = \frac{v''}{v'} = -\frac{n-1}{r} = \frac{1-n}{r}.
\end{aligned}
$$

Integrating, we get

$$
\begin{aligned}
\log v' &= (1-n)\log r + \log a \\
v' &= r^{1-n}a = \frac{a}{r^{n-1}}.
\end{aligned}
$$

Again integrating, we get

$$
v(r) = \begin{cases} b\log r + C & (n=2) \\ \frac{b}{r^(n-2)} + C & (n \geq 3), \end{cases}
$$

where b and C are constant.

Definition: The function

$$
\phi(x) = \begin{cases} -\frac{1}{2\pi}\log|x| & (n=2) \\ \frac{1}{n(n-2)\alpha(n)}\frac{1}{|x|^(n-2)} & (n \geq 3), \end{cases}
$$

defined for $x \in \mathbb{R}^n, x \neq 0$, is the fundamental solution of Laplace's equation $\triangle u = 0$ where $\alpha(n)$ is the volume of unit sphere in n- dimensional space.

Poisson's Equation :

We know that the function $x \longrightarrow \phi(x)$ is harmonic for $x \neq 0$. Now if we shift the origin to a new point y i.e $x \longrightarrow \phi(x-y)$ then this is also harmonic function of x for $x \neq y$. Let us take $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ and note that the mapping $x \longrightarrow \phi(x-y)f(y)$ $(x \neq y)$ is harmonic, where $y \in \mathbb{R}^n$ and thus so is the sum of finitely many such expression built for different points of y.
This reasoning might suggest that the convolution

$$
\begin{aligned}
u(x) &= \int_{\mathbb{R}^n} \phi(x-y)f(y)dy \\
&= \begin{cases} -\frac{1}{2\pi}\int_{\mathbb{R}^n}\log(|x-y|)f(y)dy & (n=2) \\ \frac{1}{n(n-2)\alpha(n)}\int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{(n-2)}}dy & (n \geq 3), \end{cases}
\end{aligned}
$$

will solve Laplace's equation $\triangle u = 0$. However, this is wrong. We cannot just compute

$$\triangle u = \int_{\mathbb{R}^n} \triangle_x \phi(x-y) f(y) dy = 0.$$

Indeed, as $D^2\phi(x-y)$ is not summable near the singularity at $y = x$ and so the differentiation under the integral sign is incorrect and unjustified ?
Now we show this in the case of one-dimensional Laplace equation.
In one-dimensional case we have $\phi(x) = |x|$ as $ax + b$ will be general solution of $\frac{d^2u}{dx^2} = 0$.
Now let us define

$$u(x) = \int_{-\infty}^{\infty} |x-y| f(y) dy, \tag{1}$$

with as nice a function $f(y)$ as we wish.
If the second derivative of $|x|$ vanishes and if we can differentiate under the integral sign in (1), then we should have $u^{''}(x) = 0$. For the sake of simplicity, we assume that the function $f(y)$ is smooth and vanishes outside of a finite interval so that all differentiations under the integral sign are justified:

$$\begin{aligned} u'(x) &= \frac{d}{dx}\int_{-\infty}^{\infty} |x-y| f(y) dy = \frac{d}{dx}\int_{-\infty}^{x} (x-y) f(y) dy + \frac{d}{dx}\int_{x}^{\infty} (y-x) f(y) dy \\ &= \int_{-\infty}^{x} f(y) dy - \int_{x}^{\infty} f(y) dy \end{aligned}$$

Now again differentiating w.r.to x, we get

$$u^{''}(x) = f(x) + f(x) = 2f(x), \tag{2}$$

Therefore, the function $u(x)$ is not a solution of the Laplace equation but rather of the Poisson's equation with the right side given by function $(-2f(x))$. In order to get rid of the factor (-2), we introduce

$$\phi_1(x) = -\frac{1}{2}|x|, \quad x \in R$$

and observe that for any "nice" function f, the function

$$u(x) = \int_{-\infty}^{\infty} \phi_1(x-y) f(y) dy, \quad x \in R$$

is the solution of the Poisson's equation

$$-u^{''}(x) = f(x), \quad x \in R.$$

Now solution of the Poisson's equation

$$- \triangle u = f. \tag{3}$$

$$
\begin{aligned}
\text{Let,} \quad u(x) &= \int_{\mathbb{R}^n} \phi(x-y) f(y) dy \\
&= \begin{cases} -\frac{1}{2\pi} \int_{\mathbb{R}^2} \log(|x-y|) f(y) dy & (n=2) \\ \\ \frac{1}{n(n-2)\alpha(n)} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^(n-2)} dy & (n \geq 3), \end{cases}
\end{aligned} \tag{4}
$$

Theorem: Define u by (4). Then (i) $u \in C^2(\mathbb{R}^n)$ *and* (ii) $-\triangle u = f$ *in* $\mathbb{R}^n$. Here for simplicity we will assume that $f \in C_c^2(\mathbb{R}^n) i.e.$ f is twice continuously differentiable, with compact support. Now we can see that (4) provides a formula for a solution of Poisson's equation (3) in $\mathbb{R}^n$.

Proof: (i) We have

$$
\begin{aligned}
u(x) &= \int_{\mathbb{R}^n} \phi(x-y) f(y) dy \\
&= \int_{\mathbb{R}^n} \phi(y) f(x-y) dy
\end{aligned}
$$

$$
\frac{u(x+he_i)-u(x)}{h} = \int_{\mathbb{R}^n} \phi(y) \left[\frac{f(x+he_i-y)-f(x-y)}{h} \right] f(y) dy \tag{5}
$$

where $h \neq 0$ and $e_i = (0, ..., 1, ..., 0)$, the 1 is in the i^{th} slot i.e. the unit vector in the direction of x_i.

Now we have

$$
\frac{f(x+he_i-y)-f(x-y)}{h} \longrightarrow \frac{\partial f(x-y)}{\partial x_i} \; as \; h \longrightarrow 0
$$

uniformly in $y \in \mathbb{R}^n$ (now here we will use the fact that f is compactly supported). Now we may pass to the limit $h \longrightarrow 0$ in (5) we get

$$
\frac{\partial u}{\partial x_i} = \int_{\mathbb{R}^n} \phi(y) \frac{\partial f(x-y)}{\partial x_i} dy \qquad \text{for } (i = 1, 2, ..., n)
$$

A very similar argument shows that

$$
\frac{\partial^2 u}{\partial x_i \partial x_j} = \int_{\mathbb{R}^n} \phi(y) \frac{\partial^2 f(x-y)}{\partial x_i \partial x_j} dy. \tag{6}
$$

As the expression on the right hand side of (6) is continuous in the variable x, so we have $u \in C^2(\mathbb{R}^n)$.

(ii) Now we show that $u(x)$ satisfied the Poisson's equation. From above we know that

$$
\triangle u(x) = \int_{\mathbb{R}^n} \phi(y) \triangle_x f(x-y) dy.
$$

Since $\phi(y)$ has singularity at $y = 0$, so we take a small $\epsilon > 0$ (that we will send to zero at the end of the proof) and split the integral above into the integral over the ball $B(o, \epsilon)$ of radius ϵ centered at $y = 0$ and its complement.

$$
\begin{aligned}
\triangle u(x) &= \int_{B(o,\epsilon)} \phi(y) \triangle_x f(x-y) dy + \int_{\mathbb{R}^n - B(o,\epsilon)} \phi(y) \triangle_x f(x-y) dy \\
&= I_\epsilon(x) + J_\epsilon(x).
\end{aligned}
$$

Since this decomposition holds for any $\epsilon > 0$. Therefore we have

$$\triangle u(x) = \lim_{\epsilon \longrightarrow 0} (I_\epsilon(x) + J_\epsilon(x)). \tag{7}$$

Now we compute the limit in the right hand side of (7) in order to verify the Poisson's equation. Now

$$I_\epsilon(x) = \int_{B(o,\epsilon)} \phi(y) \triangle_x f(x-y)dy$$

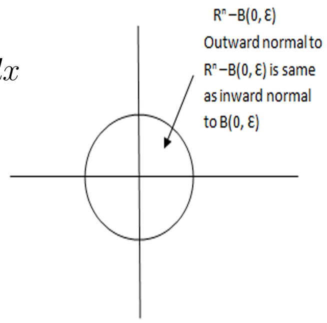
$$\begin{aligned}
\text{So,} \quad |I_\epsilon(x)| &= |\int_{B(o,\epsilon)} \phi(y) \triangle_x f(x-y)dy| \\
&\leq \int_{B(o,\epsilon)} |\phi(y)|| \triangle_x f(x-y)|dy \\
&\leq \|D^2 f\|_{L^\infty(\mathbb{R}^n)} \int_{B(o,\epsilon)} |\phi(y)|dy, \\
\text{since} \quad | \triangle_x f(x-y)| &\leq \|D^2 f\|_{L^\infty(\mathbb{R}^n)}. \\
\text{Now for n} = 2, \quad \int_{B(o,\epsilon)} |\phi(y)|dy &= \frac{1}{2\pi} \int_o^\epsilon \int_o^{2\pi} |(\log r)| r dr dw \\
&= \frac{1}{2\pi} 2\pi \int_o^\epsilon |(\log r)| r dr \\
&= [|\log r| \frac{r^2}{2}]_0^\epsilon + \int_o^\epsilon \frac{1}{r}\frac{r^2}{2} dr \text{ (we assume that } 0 < \epsilon < 1) \\
&\leq \epsilon^2 |\log \epsilon| \\
\text{So for n} = 2, \quad |I_\epsilon(x)| &\leq \|D^2 f\|_{L^\infty(\mathbb{R}^n)} \epsilon^2 |\log \epsilon| \\
\text{For } n \geq 3, \quad \int_{B(o,\epsilon)} |\phi(y)|dy &\leq \int_{B(o,\epsilon)} \frac{1}{n(n-2)\alpha(n)} \frac{1}{y^(n-2)} dy \\
&= \frac{1}{n(n-2)\alpha(n)} \int_o^\epsilon \int_{S^{(n-1)}} \frac{r^{(n-1)}}{r^{(n-2)}} dr dw \\
&= \frac{1}{n(n-2)\alpha(n)} \int_o^\epsilon \int_{S^{(n-1)}} r dr dw \leq \epsilon^2 \\
\text{So} \quad |I_\epsilon(x)| &\leq \|D^2 f\|_{L^\infty(\mathbb{R}^n)} \int_{B(o,\epsilon)} |\phi(y)|dy \\
&\leq \begin{cases} C\epsilon^2 |\log \epsilon| & (n = 2) \\ \\ C\epsilon^2 & (n \geq 3), \end{cases}
\end{aligned}$$

From above we can conclude that $\lim_{\epsilon \longrightarrow 0} I_\epsilon(x) = 0$, uniformly in $x \in \mathbb{R}^n$.
Therefore the contribution to $\Delta u(x)$ comes from $J_\epsilon(x)$.

$$\begin{aligned}
\text{Now} \quad J_\epsilon(x) &= \int_{\mathbb{R}^n - B(o,\epsilon)} \phi(y) \triangle_x f(x-y)dy \\
\text{Now we know that,} \quad \triangle_x f(x-y) &= \triangle_y f(x-y) \\
J_\epsilon(x) &= \int_{\mathbb{R}^n - B(o,\epsilon)} \phi(y) \triangle_y f(x-y)dy
\end{aligned}$$

Gauss Green's Formula: Let $v(x)$ be a vector valued function and $f(x)$ is a scalar valued function over a nice domain U. Then

$$\int_U v(x) gradf(x)dx = \int_{\partial U} (v(x).\nu) f(x) dS(x) - \int_U f(x) divv(x) dx$$



$$\triangle f = div(gradf)$$

Choose $f = \phi, \quad v(x) = D_y f = gradf$

$$\int_U D_y f.D\phi dx = \int_{\partial U} (gradf.\nu)\phi ds - \int_U \phi div(D_y f) dv$$

$$\begin{aligned}
J_\epsilon(x) &= \int_{R^n - B(o,\epsilon)} \phi(y) \triangle_y f(x-y) dy \\
&= \int_{R^n - B(o,\epsilon)} \phi(y) div(D_y f(x-y)) dy \\
&= -\int_{R^n - B(o,\epsilon)} D\phi(y).D_y f(x-y) dy + \int_{\partial B(o,\epsilon)} \phi(y) \frac{\partial f}{\partial \nu}(x-y) dS(y) \\
&= K_\epsilon + L_\epsilon
\end{aligned}$$

ν denoting the inward pointing unit normal along $\partial B(o,\epsilon)$

$$Now, \quad L_\epsilon = \int_{\partial B(o,\epsilon)} \phi(y) \frac{\partial f}{\partial \nu}(x-y) dS(y).$$

$$\begin{aligned}
|L_\epsilon| &\leq \|Df\|_{L^\infty(R^n)} \int_{\partial B(o,\epsilon)} |\phi(y)| dS(y) \\
&\leq \begin{cases} C\epsilon |\log \epsilon| & (n=2) \\ C\epsilon & (n \geq 3), \end{cases}
\end{aligned}$$

$$\begin{aligned}
\text{for} \quad n=2, \quad \int_{\partial B(o,\epsilon)} |\phi(y)| dS(y) &= \int_{\partial B(o,\epsilon)} \frac{1}{2\pi} |\log \epsilon| dS(y) \\
&= \frac{1}{2\pi} |\log \epsilon| 2\pi\epsilon = \epsilon |\log \epsilon| \\
\text{for} \quad n \geq 3, \quad \int_{\partial B(o,\epsilon)} |\phi(y)| dS(y) &= \int_{\partial B(o,\epsilon)} \frac{1}{n(n-2)\alpha(n)} \frac{1}{\epsilon^{(n-2)}} dS(y) \\
&= \frac{1}{n(n-2)\alpha(n)} \frac{1}{\epsilon^{(n-2)}} n\alpha(n) \epsilon^{(n-1)} \leq \epsilon \\
\text{So in both cases, we have} \quad \lim_{\epsilon \longrightarrow 0} L_\epsilon &= 0.
\end{aligned}$$

So the main contribution to $\Delta u(x)$ must come from K_ϵ.

$$K_\epsilon = -\int_{R^n - B(0,\epsilon)} D\phi(y).D_y f(x-y)dy.$$

Now integrating by parts using Green's formula once again, we get

$$K_\epsilon = \int_{R^n - B(0,\epsilon)} (\triangle\phi(y))f(x-y)dy - \int_{\partial B(0,\epsilon)} \frac{\partial\phi(y)}{\partial\nu} f(x-y)dS(y).$$

$$\text{Now,} \quad \int_{R^n - B(0,\epsilon)} \triangle\phi(y)f(x-y)dy = 0.$$

$$\text{As} \quad \triangle\,\phi(y) = 0, \text{ since } \; y \in R^n - B(0,\epsilon) \; \text{ so } y \neq 0.$$

$$\text{Now,} \quad K_\epsilon = 0 - \int_{\partial B(0,\epsilon)} \frac{\partial\phi(y)}{\partial\nu} f(x-y)dS(y).$$

Now consider the case $n \geq 3$, then

$$\begin{aligned}
D\phi(y) &= \frac{1}{n(n-2)\alpha(n)} D(|y|^{-(n-2)}) \\
&= \frac{1}{n(n-2)\alpha(n)} \left(\frac{-n+2}{2}\right) (y_1^2 + y_2^2 + \;\ldots\; + y_n^2)^{\frac{-n+2}{2}-1} 2y \\
&= -\frac{1}{n(n-2)\alpha(n)} (n-2)\,|y|^{-n} y \\
&= -\frac{1}{n\alpha(n)}.\frac{y}{|y|^n} \;\; (y \neq 0)
\end{aligned}$$

$$\begin{aligned}
\text{and} \quad \nu = \frac{-y}{|y|} &= -\frac{y}{\epsilon} \;\text{ on }\; \partial B(0,\epsilon). \\
\text{Consequently,} \quad \frac{\partial\phi(y)}{\partial\nu} &= \nu.D\phi(y) \\
&= -\left(\frac{y}{\epsilon}\right).\left(\frac{-1}{n\alpha(n)}\frac{y}{\epsilon^n}\right) \\
&= \frac{1}{n\alpha(n)\epsilon^{n-1}} \;\text{ on }\; \partial B(0,\epsilon)
\end{aligned}$$

Since $n\alpha(n)$ is the surface area of the sphere $\partial B(0,\epsilon)$, we have

$$\begin{aligned}
K_\epsilon &= -\frac{1}{n\alpha(n)\epsilon^{n-1}} \int_{\partial B(o,\epsilon)} f(x-y)dS(y) \\
&= -\frac{1}{n\alpha(n)\epsilon^{n-1}} \int_{\partial B(x,\epsilon)} f(y)dS(y) \\
&= -\frac{1}{|\partial B(x,\epsilon)|} \int_{\partial B(x,\epsilon)} f(y)dS(y) \\
&= -\fint_{\partial B(x,\epsilon)} f(y)dS(y).
\end{aligned}$$

$$\begin{aligned}
\text{Thus} \quad \lim_{\epsilon\to 0} K_\epsilon(x) &= -f(x) \\
\text{so} \quad \triangle\,u(x) &= -f(x). \text{ Hence proved.}
\end{aligned}$$

Note: If $f(x)$ is continuous at x then $\oint_{\partial B(x,r)} f(y)dS(y) = f(x)$ as $r \to 0$.
Now $\oint_{\partial B(x,r)} f(y)dS(y) - f(x = \oint_{\partial B(x,r)}(f(y) - f(x))dS(y)$.
Now choose $r < \delta$, and using the continuity of f at x, we get

$$|\oint_{\partial B(x,r)} f(y)dS(y) - f(x)| < \oint_{\partial B(x,r)} |f(y) - f(x)|dS(y) < \epsilon,$$

hence proves.

Mean Value Formula:

Now we will show that locally $u(x)$ is close to its average. Intuitively this implies that $u(x)$ can not behave very irregularly and should have limited room to oscillate.
Now let us first look mean value property in one-dimension. Any harmonic function in one-dimension is linear $u(x) = ax + b$, and then of course, for any $x \in \mathbb{R}$ and any $l > 0$, we have

$$u(x) = \frac{1}{2}\left(u(x+l) + u(x-l)\right) = \frac{1}{2l}\int_{x-l}^{x+l} u(y)dy.$$

Now following result is nothing but generalization to harmonic function in higher dimensions.

Theorem: Let $U \subset \mathbb{R}^n$ be an open set and let $B(x,r)$ be a ball centered at $x \in \mathbb{R}^n$ of radius $r > 0$ contained in U. Assumed that the function $u(x)$ satisfies $\triangle u = 0$ for all $x \in U$ and that $u \in C^2(U)$. Then we have

$$\begin{aligned} u(x) &= \frac{1}{|B(x,r)|}\int_{B(x,r)} u(y)dy = \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} u(t)dS(y) \\ &= \oint_{B(x,r)} u(y)dy = \oint_{\partial B(x,r)} u(y)dS(y). \end{aligned}$$

Proof: Let us fix the point $x \in U$ and define

$$\phi(r) = \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} u(y)dS(y) \qquad (1)$$

It is easy to note that since $u(x)$ is continuous, we have

$$\lim_{r \longrightarrow 0} \phi(r) = u(x).$$

Therefore our proof will be over if we will prove that

$$\phi'(r) = \frac{d\phi}{dr} = 0 \text{ for all } r > 0.$$

For this, we will use the polar coordinates i.e. $y = x + rz$ with $z \in \partial B(0,1)$. Now equation (1) will take the following form

$$\phi(r) = \frac{1}{|\partial B(0,1)|}\int_{\partial B(0,1)} u(x+rz)dS(z).$$

Now differentiating above w.r.t.'r', we get

$$\phi'(r) = \frac{1}{|\partial B(0,1)|}\int_{\partial B(0,1)} Du(x+rz)\cdot z dS(z).$$

Going back to $y-$ variable, we get

$$\begin{aligned}\phi'(r) &= \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} Du(y)\cdot\frac{y-x}{r}dS(y)\\ &= \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)}\frac{\partial u}{\partial \nu}dS(y) \qquad (2)\end{aligned}$$

where $\frac{y-x}{r}$ is unit vector on $\partial B(x,r)$ and $\frac{\partial u}{\partial \nu}$ is the directional derivative.
Now we will use the following Green's formula

$$\int_U f\triangle g dy = \int_{\partial U} f\frac{\partial g}{\partial \nu}dS - \int_U Df.Dgdy$$

by taking $f = 1$ and $g = u$, we get

$$\int_{B(x,r)}\triangle u dy = \int_{\partial B(x,r)}\frac{\partial u}{\partial \nu}dS(y) \qquad (3)$$

Now combining equation (2) and (3), we get

$$\begin{aligned}\phi'(r) &= \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)}\frac{\partial u}{\partial \nu}dS(y)\\ &= \frac{1}{|B(x,r)|}\int_{\partial B(x,r)}\triangle u(y)dS(y) = 0(\because u \text{ is harmonic})\\ \text{or}\quad \phi'(r) &= \frac{|B(x,r)|}{|\partial B(x,r)|}\frac{1}{|B(x,r)|}\int_{B(x,r)}\triangle u(y)d(y)\\ &= \frac{r}{n}⨏_{B(x,r)}\triangle u y = 0.\end{aligned}$$

This implies that $\phi(r)$ is a constant.

$$\begin{aligned}\text{So,}\quad \phi(r) &= \lim_{t\longrightarrow 0}\frac{1}{|\partial B(x,t)|}\int_{\partial B(x,t)} u(y)dS(y) = u(x)\\ \text{So,}\quad u(x) &= \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} u(y)dS(y)\end{aligned}$$

In order to prove the first equality we use the polar coordinates once again

$$\begin{aligned}\frac{1}{|B(x,r)|}\int_{B(x,r)} u(y)dy &= \frac{1}{|B(x,r)|}\int_0^r\left(\int_{\partial B(x,s)} u(y)dS(y)\right)ds\\ &= \frac{1}{|B(x,r)|}\int_0^r u(x)n\alpha(n)s^{n-1}ds\\ &= u(x)\frac{n\alpha(n)r^n}{n|B(x,r)|} = \frac{u(x)}{n}\frac{n\alpha(n)r^n}{\alpha(n)r^n} = u(x)\\ \text{i.e.,}\quad u(x) &= \frac{1}{|B(x,r)|}\int_{B(x,r)} u(y)dy = ⨏_{B(x,r)} u(y)dy.\end{aligned}$$

Theorem: (Converse to mean value property)
If $u \in C^2(U)$ satisfied

$$u(x) = \fint_{\partial B(x,r)} u dS = \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} u dS$$

for each ball $B(x,r) \subset U$, then u is harmonic.
Proof: We will try to prove it by contradiction.
Assume that $\triangle u \neq 0$, then there exists some ball $B(x,r) \subset U$, such that say $\triangle u > 0$, within $B(x,r)$.
Now since $u(x) = \fint_{\partial B(x,r)} u(y) dS(y)$. This says that $u(x)$ represents the average of u over the surface of the ball $B(x,r)$.
Now define $\phi(r)$ as above i.e.

$$\phi(r) = \fint_{\partial B(x,r)} u(y) dS(y).$$

Since $u(x)$ is average over the surface of the ball $B(x,r)$ so we must have

$$\phi'(r) = 0$$

$$\begin{aligned} \text{Now} \quad 0 = \phi'(r) &= \frac{r}{n} \fint_{B(x,r)} \triangle u(y) dy > 0 \quad (\text{as} \quad \triangle\, u > 0 \text{ in } B(x,r)) \\ \text{i.e. } \; 0 &> 0. \\ \text{This is } & \text{ a } \quad \text{contradiction. Hence the statement of theorem is true.} \end{aligned}$$

Strong Maximum Principle:

Theorem: Suppose $u \in C^2(U) \bigcap C(\overline{U})$ is harmonic within U then

$$(i) \; \max_{\tilde{U}} u = \max_{\partial U} u.$$

(ii) Furthermore, if U is connected and there exist a point $x_0 \in U$ such that

$$u(x_0) = \max_{\tilde{U}} u.$$

Then u is constant with U. Assertion (i) is maximum principle for Laplace equation and (ii) Strong Maximum Principle.

Proof: We will prove (ii) from which (i) follows;
By assumption, u is bounded from above and attains its maximum in U at a point x_0.
Let

$$u(x_0) = M = \max_{\overline{U}}\{u\}$$

and consider the following set

$$F = u^{-1}\{M\} = \{x \in U : \; u(x) = M\}.$$

The following fact may be noted
(i) Since singleton sets are closed in usual topology and so $\{M\}$ is closed. Now continuity of u implies that $u^{-1}\{M\} = F$ is closed.
(ii) F is also open.
Now for any $0 < r < \text{dist}(x_0, \partial U)$

$$M = u(x_0) = \fint_{B(x_0,r)} u(y)dy \leq M$$

The equality holds only when $u(y) = M \ \forall\ y \in B(x_0, r)$ otherwise if $u(y) < M$ for some $y \in B(x_0, r)$, then we will get a contradiction and so $u(y) = M \ \forall\ y \in B(x_0, r)$.
This implies that $B(x_0, r) \subseteq F$ and so F is open.
Now the subset F of U is closed as well as open and keeping in mind the connectedness of U, we get $F = U$, i.e. $u(x) = M \ \forall\ x \in U$. Hence proved.

Now we use (ii) to prove (i).
Certainly

$$\max_{\overline{U}} u \geq \max_{\partial U} u \text{ since } \partial U \subseteq U.$$

Now assume $\max_{\overline{U}} u > \max_{\partial U} u$, then the maximum is achieved at some interior point x_0.
Let U_{x_0} be the connected component containing x_0. By (ii), we know that $u \equiv$ constant $= u(x_0)$ in U_{x_0}.

$$\text{Since } u \in C(\overline{U}), \text{ we know that } \lim_{x \to y \in \partial U_{x_0}} u(x) = u(x_0) = u(y). \text{ Contradicting}$$
$$\max_{\partial U} u < \max_{\overline{U}} u.$$

Cor 1: Assume that U is a connected domain, and u solves

$$\begin{aligned} \Delta u &= 0 \quad \text{in } U \\ u &= g \text{ on } \partial U. \qquad (1) \end{aligned}$$

Assume in addition, that $g \geq 0$, g is continuous on ∂U, and $g(x) \neq 0$. Then $u(x) > 0$ at all $x \in U$.
Proof. The proof of this corollary immediately follows from minimum principle: $\min_{x \in U} u(x) \geq 0$, and u can not attain its minimum inside U, thus $u(x) > 0$ for all $x \in U$.
Cor 2: (Uniquness) Let g be continuous on ∂U and f be continuous in U. Then there exists at most one solution $u \in C^2(U) \bigcap C(\bar{U})$ to the boundary value problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } U \\ u &= g \text{ on } \partial U. \qquad (2) \end{aligned}$$

Proof: Let u_1 and u_2 be two solution of (2). Then the difference $w = u_1 - u_2$ satifies the homogeneous problem

$$\begin{aligned} \Delta w &= 0 \quad \text{in } U \\ w &= 0 \text{ on } \partial U. \qquad (3) \end{aligned}$$

The maximum principle implies that $w \leq 0$ in U while the minimum principle implies that $w \geq 0$ in U, whence $w \equiv 0$, and thus $u_1 = u_2$ in U, proving the corollary.

Regularity of harmonic functions: Now we prove that, if $u \in C^2(U)$ is harmonic, then $u \in C^\infty$. This sort of assertion is called a regularity theorem. The interesting point is that the algebraic structure of Laplace's equation $\triangle u = \sum_{i=1}^{n} u_{x_i x_i}$ leads to the analytic deduction that all the partial derivatives of u exists even those which do not appear in PDE.

Theorem: If $u \in C(U)$ satisfies the mean value property for each ball $B(x, r) \subset U$. Then $U \in C^\infty(U)$

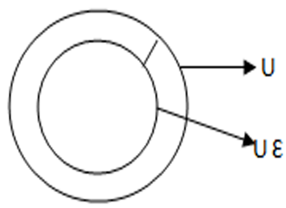
Or

If $u \in C^2(U)$ be a harmonic function in a domain U. Then u is infinitely differentiable in U.
Proof: The proof of this theorem comes via a miracle. We first define a "smoothed" version of u, and then verify that the "smoothed" version coincides with the original, hence original is also infinitely smooth.
Consider a radial non-negative functions $\eta(x) \geq 0$ that depends only on $|x|$ such that

(i) $\eta(x) = 0$ for $|x| \geq 1$
(ii) $\eta(x)$ is infinitely differentiable and
(iii) $\int_{\mathbb{R}^n} \eta(x) dx = 1$.

Also for each $\epsilon \in (0, 1)$ define its stretched version

$$\eta_\epsilon(x) = \frac{1}{\epsilon^n} \eta\left(\frac{x}{\epsilon}\right).$$



It is easy to verify that η_ϵ satisfied all the above three conditions. Moreover the function

$$u^\epsilon(x) = \int_U \eta_\epsilon(x - y) u(y) dy \qquad (1)$$

is infinitely differentiable in the slightly smaller domain

$$U_\epsilon = \{x \in U | \text{dist}(x, \partial U) > \epsilon\}.$$

The reason is that we can differentiate infinitely many times under the integral sign in equation (1).

Our main claim is that, because of the mean value property,

$$u^\epsilon(x) = u(x) \text{ for all } x \in U_\epsilon. \qquad (2)$$

This will immediately imply that $u(x)$ is infinitely differentiable in the domain U_ϵ and as any point x from U lies in U_ϵ if $\epsilon < \text{dist}(x, \partial U)$, it follows that $u(x)$ is infinitely

differentiable at all points $x \in U$.
Let us now verify equation (2)

$$\begin{aligned} u^\epsilon(x) &= \int_U \eta_\epsilon(x-y)u(y)dy \\ &= \frac{1}{\epsilon^n}\int_U \eta\left(\frac{|x-y|}{\epsilon}\right)u(y)dy \\ &= \frac{1}{\epsilon^n}\int_{B(x,\epsilon)} \eta\left(\frac{|x-y|}{\epsilon}\right)u(y)dy. \end{aligned}$$

The last equality holds because $\eta(z)=0$ if $|z|\geq 1$, where $\eta_\epsilon(z)=0$ if $|z|\geq\epsilon$.
Changing variables $y = x+\epsilon z$ gives

$$u^\epsilon(x) = \frac{1}{\epsilon^n}\int_{B(0,1)} \eta\left(\frac{\epsilon z}{\epsilon}\right)u(x+\epsilon z)dz$$

$$u^\epsilon(x) = \int_0^1 \eta(r)\left[\int_{\partial B(0,1)} u(x+\epsilon rw)dS(w)\right]r^{n-1}dr \qquad (3)$$

The mean value property implies that

$$\int_{\partial B(0,1)} u(x+\epsilon rw)dS(w) = u(x)|\partial B(0,1)|$$

using this in (3), we get

$$\begin{aligned} u^\epsilon(x) &= u(x)\int_0^1 \eta(r)|\partial B(0,1)|r^{n-1}dr \\ &= u(x)\int_{B(0,1)} \eta(y)dy = u(x). \end{aligned}$$

Thus $u^\epsilon \equiv u$ in U_ϵ, and $u \in C^\infty(U_\epsilon)$ for each $\epsilon > 0$.

Estimates on derivatives:

Theorem: Let $u(x)$ be a harmonic function in a domain U and let $B(x_0, r)$ be a ball contained in U centered at a point $x_0 \in U$. Then there exists universal constants C_n and D_n that depends only on the dimension n so that we have

$$u(x_0) \leq \frac{C_n}{r^n}\int_{B(x_0,r)} |u(y)|dy \qquad (1)$$

and

$$|Du(x_0)| \leq \frac{D_n}{r^{n+1}}\int_{B(x_0,r)} |u(y)|dy \qquad (2)$$

Proof: The first estimate follows immediately from the mean value formula. Now for the derivation of second inequality, we note that if $u(x)$ is harmonic then so are the partial derivatives $\frac{\partial u}{\partial x_j}$,

$$\begin{aligned}\frac{\partial u(x_0)}{\partial x_j} &\leq \frac{1}{|B(x_0,\frac{r}{2})|}\left|\int_{B(x_0,\frac{r}{2})}\frac{\partial u(y)}{\partial x_j}dy\right| \\ &= \frac{1}{|B(x_0,\frac{r}{2})|}\left|\int_{\partial B(x_0,\frac{r}{2})}u(y)\nu_j(y)dS(y)\right| \text{ (By Gauss Green's formula)}, \qquad (3)\end{aligned}$$

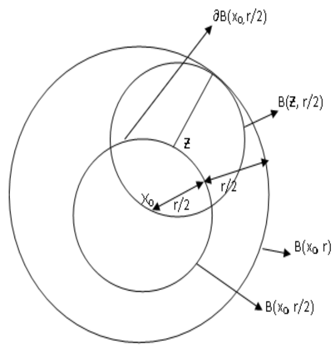
where, $\nu_j(y)$ is the $j-$ th component of the outward normal. Continuing this, we see that

$$\begin{aligned}\frac{\partial u(x_0)}{\partial x_j} &\leq \frac{2^n}{\alpha(n)r^n}.\frac{n\alpha(n)r^{n-1}}{2^{n-1}}\sup_{z\in\partial B(x_0,\frac{r}{2})}|u(z)| \\ &= \frac{2n}{r}\sup_{z\in\partial B(x_0,\frac{r}{2})}|u(z)|. \qquad (4)\end{aligned}$$

Now, we can use the estimate (1) applied at any point $z\in\partial B(x_0,\frac{r}{2})$:

$$|u(z)| \leq \frac{C_n}{(\frac{r}{2})^n}\int_{B(x_0,\frac{r}{2})}|u(z')|dz'. \qquad (5)$$

However, since $|x_0 - z| \leq \frac{r}{2}$ (this is why, we took a smaller ball in (3)), any such ball $B(z,\frac{r}{2})$ is contained inside the ball $B(x_0,r)$, thus (5) implies that



$$|u(z)| \leq \frac{C_n}{(\frac{r}{2})^n}\int_{B(x_0,r)}|u(z')|dz'.$$

Now it follows from (4) that

$$\begin{aligned}\left|\frac{\partial u(x_0)}{\partial x_j}\right| &\leq \frac{2n}{r}\frac{C_n}{(\frac{r}{2})^n}\int_{B(x_0,r)}|u(z')|dz' \\ &= \frac{D_n}{r^{n+1}}\int_{B(x_0,r)}|u(y)|dy. \qquad (6)\end{aligned}$$

which proves the result (2).

Now we proceed to find the estimates of higher order derivatives. We have proved that theorem is true for $K = 0$ and $K = 1$.

$$|u(x)| \leq \frac{1}{\alpha(n)}(\frac{2}{r})^n\|u\|_{L^1B(x_0,r)}$$

and

$$|u_{x_i}(x_0)| \leq \frac{2^{n+1}n}{\alpha(n)}\frac{1}{r^{n+1}}\|u\|_{L^1B(x_0,r)}$$

Assume now $K \geq 2$ and theorem is true for all balls in U and each multi index of order less than or equal to $K-1$. Fix $B(x_0, r) \subset U$ and let α be a multi index with $|\alpha| = K$. Then $D^\alpha u = (D^\beta u)_{x_i}$ for some $i \in \{1, ..., n\}$, $\|\beta\| = K-1$.
Now as previously, we can easily prove that

$$|D^\alpha u(x_0)| \leq \frac{nK}{r} \|D^\beta u\|_{L^\infty(\partial B(x_0, \frac{r}{K}))}$$

If $x \in \partial B(x_0, \frac{r}{K})$, then $B(x, \frac{K-1}{K} r) \subset B(x_0, r) \subset U$.
Thus we can apply the theorem for $K-1$, we get

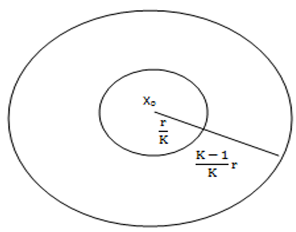
$$\|D^\beta u_x\| \leq \frac{(2^{n+1} n(K-1))^{K-1}}{\alpha(n)(\frac{K-1}{K} r)^{n+K-1}} \|u\|_{L^1 B(x_0, r)}.$$

Combining this with previous, we get

$$\begin{aligned}
|D^\alpha u_{x_0}| &\leq \frac{nK}{r} \frac{(2^{n+1})^{K-1} n^{K-1} (K-1)^{K-1} K^{n+K-1}}{\alpha(n)(K-1)^{n+K-1} r^{n+K-1}} \|u\|_{L^1 B(x_0, r)} \\
&= \frac{nK}{r} \frac{(2^{n+1})^{K-1} n^{K-1} K^{n+K-1}}{\alpha(n)(K-1)^n r^{n+K-1}} \|u\|_{L^1(x_0, r)} \\
&= \frac{(2^{n+1})^{K-1} n^K K^{n+K}}{\alpha(n)(K-1)^n r^{n+K}} \|u\|_{L^1 B(x_0, r)} \\
&= \frac{(2^{n+1})^K n^K K^K K^n}{\alpha(n)(K-1)^n 2^{n+1} r^{n+K}} \|u\|_{L^1 B(x_0, r)} \\
&= \frac{(2^{n+1} nK)^K}{\alpha(n) r^{n+K}} \frac{K^n}{2^{n+1}(K-1)^n} \|u\|_{L^1 B(x_0, r)} \\
&< \frac{(2^{n+1} nK)^K}{\alpha(n) r^{n+1}} \|u\|_{L^1 B(x_0, r)},
\end{aligned}$$

here, for $K \geq 2$ $\quad \dfrac{K^n}{2^{n+1}(K-1)^n} < 1.$

Liouville's Theorem: Let $u(x)$ be a harmonic bounded function in $\mathbb{R}^n$. Then $u(x)$ is equal identically to a constant.



Proof: Let us assume that $|u(x)| \leq M$ for all $x \in \mathbb{R}^n$, we fix $x_0 \in \mathbb{R}^n$ and from the theorem for local estimates, we have

$$|Du(x_0)| \leq \frac{2^{n+1} n}{\alpha(n) r^{n+1}} \|u\|_{L^1 B(x_0, r)}.$$

$$\text{Now } \|u\|_{L^1 B(x_0, r)} \leq \alpha(n) r^n M$$

$$\begin{aligned}
\text{so } |Du(x_0)| &\leq \frac{2^{n+1} n}{\alpha(n) r^{n+1}} \alpha(n) r^n M \\
&= \frac{2^{n+1} n}{r} M.
\end{aligned}$$

As this is true for any $r > 0$, we may let $r \longrightarrow \infty$ and conclude that $Du(x_0) = 0$ thus $u(x)$ is equal identically to a constant.

$$\begin{aligned} \text{Or,} \quad |Du_{x_0}| &\leq \frac{C}{r^{n+1}} \int_{B(x_0,r)} |u(y)|dy \leq \frac{C\alpha(n)r^n}{r^{n+1}} M \leq \frac{C\alpha(n)}{r} M \\ &\longrightarrow 0 \text{ as } r \longrightarrow \infty \\ \text{Thus } Du &= 0 \Longrightarrow u = \text{constant} \end{aligned}$$

Theorem : (Representation formula)
Let $f \in C_c^2(\mathbb{R}^n)$, $n \geq 3$. Then any bounded solution of $- \triangle u = f$ in $\mathbb{R}^n$ has the form $u(x) = \int_{\mathbb{R}^n} \phi(x-y)f(y)dy + C \quad (x \in \mathbb{R}^n)$ for some constant C.
Proof: Since $\phi(x)$ is defined as

$$\phi(x) = \begin{cases} -\frac{1}{2\pi} \log |x| & (n = 2) \\ \frac{1}{n(n-2)\alpha(n)} \frac{1}{|x|^(n-2)} & (n \geq 3), \end{cases}$$

Since $\phi(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$ for $n \geq 3$.
Suppose $\tilde{u}$ is another solution of the given poisson equation $-\Delta u = f$.
$\tilde{u}(x) = \int_{\mathbb{R}^n} \phi(x-y)f(y)dy$ is bounded solution of $- \triangle u = f$ in $\mathbb{R}^n$.
If u is another solution then $w = u - \tilde{u}$ is a solution of Laplace's equation $\triangle w = 0$. Now applying the Liouville's theorem, we get w is a constant so $u(x) = \int_{R^n} \phi(x-y)f(y)dy + C$.

Remark: If $n = 2$ then $\psi(x) = \frac{1}{2\pi} \log |x|$ is unbounded as $|x| \longrightarrow \infty$ and so may be $\int_{R^2} \psi(x-y)f(y)dy$.

Question: Harnack's inequality for 1-dimension:

Let us first try to understand this in one dimensional case.
Let $u(x)$ be a non-negative harmonic function on the interval $(0,1)$, that is, $u(x) = ax+b$ with some constants $a, b \in R$.
We claim that if $u(x) > 0$ for all $x \in [0,1]$ then

$$\frac{1}{3} \leq \frac{u(x)}{u(y)} \leq 3 \qquad (1)$$

for all x, y in the smaller interval $(\frac{1}{4}, \frac{3}{4})$. The constants $\frac{1}{3}$ and $\frac{3}{4}$ in equation (1) depend on the choice of the "smaller " interval. These constants will change if we will replace $(\frac{1}{4}, \frac{3}{4})$ by another subinterval of $[0,1]$. But once are fix the subinterval, they do not depend on the choice of the harmonic function. Let us now show that equation (1) holds for all $x, y \in (\frac{1}{4}, \frac{3}{4})$ without loss of generality we may assume that $x > y$. First consider the case $a > 0$. Then since $u(x)$ is increasing (because $a > 0$), we have

$$1 \leq \frac{u(x)}{u(y)} \leq \frac{u(\frac{3}{4})}{u(\frac{1}{4})} = \frac{3a+4b}{a+4b} \qquad (2)$$

As $u(x) > 0$ on $[0,1]$, we know that $b > 0$ (and $a > 0$ by assumption) using this in equation (2), we get [let $c = \frac{a}{b}$]

$$1 \leq \frac{u(x)}{u(y)} \leq \frac{3c+4}{c+4} = 3 - \frac{8}{c+4} \leq 3.$$

On the other hand, if $a < 0$ then the function u is decreasing and

$$1 \geq \frac{u(x)}{u(y)} \geq \frac{u(\frac{3}{4})}{u(\frac{1}{4})} = \frac{3c+4}{c+4} = \frac{1}{3} + \frac{8(c+1)}{3(c+4)}.$$

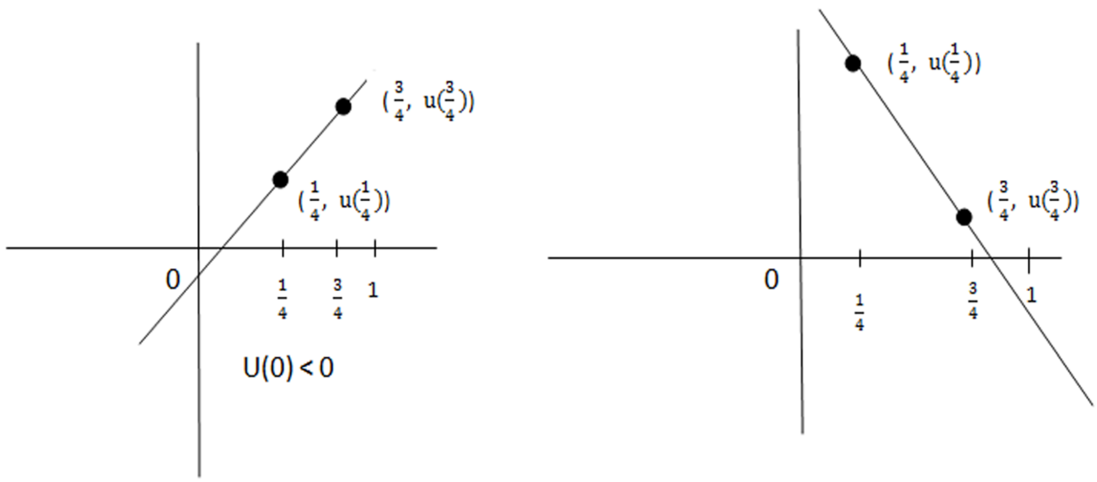
As $u(1) > 0$ we know that $a + b > 0$, and we still have $b > 0$ since $u(0) > 0$. Thus $-1 < c < 0$ and therefore

$$1 \geq \frac{u(x)}{u(y)} \geq \frac{1}{3} + \frac{8(c+1)}{3(c+4)} \geq \frac{1}{3}.$$

Now we conclude that equation (1) indeed holds i.e.

$$\frac{1}{3} \leq \frac{u(x)}{u(y)} \leq 3 \ \text{ for } \ x, y \in (\frac{1}{4}, \frac{3}{4}).$$

Geometrically equation (1) expresses the following fact:



If $u(\frac{3}{4}) >> u(\frac{1}{4})$ then the slope of the straight line connecting the points $(\frac{1}{4}, u(\frac{1}{4}))$ and $(\frac{3}{4}, u(\frac{3}{4}))$ is too large so that it would go below the $x-$ axis at $x = 0$. On the other hand if $u(\frac{1}{4}) >> u(\frac{3}{4})$ then this line would go below that $x-$ axis at $x = 1$. Therefore the condition that $u(x) > 0$ on the larger interval $[0,1]$ is very important here.

<u>**Harnack's inequality:**</u> For each connected open set $V \subset\subset U$, there exists a positive constant C, depending only on V , such that

$$\sup_V u \leq C \inf_V u$$

for all nonnegative harmonic functions u in U.
Thus in particular

$$\frac{1}{C}u(y) \leq u(x) \leq Cu(y). \quad \text{for all points } \ x, y \in V$$

Explanation: Let $x, y \in V$, then

$$u(x) \leq \sup_{\overline{V}} u \leq C \inf_{\overline{V}} u \leq Cu(y)$$

$$\therefore \forall x, y \in V, u(x) \leq Cu(y)$$

$$\text{and } u(y) \leq Cu(x) \Rightarrow \frac{1}{C}u(y) \leq u(x)$$

$$\text{So } \frac{1}{C}u(y) \leq u(x) \leq Cu(y) \ \ \forall\ x, y \in V$$

Proof: Let $r > 0$ and $r = \frac{1}{4}dist(V, \partial U)$
Let $x, y \in V$ such that $|x - y| \leq r$
Then $u(x) = \oint_{B(x,2r)} u(z)dz = \dfrac{\int_{B(x,2r)} u(z)dz}{\alpha(n)(2r)^n}$
Let $z \in B(y, r)$ then $|z - y| < r$

$$|z - x| \leq |z - y| + |y - x| < 2r$$

$$\Rightarrow z \in B(x, 2r)$$

$$\text{So } \ B(y, r) \subseteq B(x, 2r)$$

Since $u(z)$ is non-negative and $B(y, r) \subseteq B(x, 2r)$
So

$$\begin{aligned} u(x) = \frac{\int_{B(x,2r)} u(z)dz}{\alpha(n)(2r)^n} \quad &\geq \quad \frac{\int_{B(y,r)} u(z)dz}{\alpha(n)(2r)^n} \\ &= \quad \frac{\oint_{B(y,r)} u(z)dz\alpha(n)r^n}{\alpha(n)(2r)^n} = \frac{\oint_{B(y,r)} u(z)dz}{2^n} \\ &= \quad \frac{1}{2^n}u(y) \end{aligned}$$

$$\therefore \forall\ x, y \in V \text{ such that } \ |x - y| < r$$

$$u(x) \geq \frac{1}{2^n}u(y)$$

and

$$u(y) \geq \frac{1}{2^n}u(x)$$

$$\frac{1}{2^n}u(y) \leq u(x) \leq 2^n u(y)$$

Let $U_{x\in\overline{V}}B^0(x, \frac{r}{2})$ be an open covering of $\overline{V}$.
As $\overline{V}$ is compact, $\exists\ x_1, x_2, ..., x_N \in \overline{V}$ such that

$$\overline{V} \subseteq B^0(x_1, \frac{r}{2}) \cup ... \cup B^0(x_N, \frac{r}{2})$$

$$\text{write } A_i = \overline{V} \cap B^0(x_i, \frac{r}{2}), i = 1, 2, ..., N$$

Then as V is connected ($\therefore\ \overline{V}$ is connected).

Claim: Each A_i $(i = 1, 2, ..., N)$ must intersect with some $A_j(i \neq j)(j = 1, 2, ..., N)$. If not then there exists $A_i(i = 1, 2, ..., N)$ such that $A_i \cap A_j = \phi\ \ \forall\ j(j = 1, 2, ..., N)$ $(i \neq j)$. Then

$$\overline{V} = A_i \cup (\cup_{j=1,i\neq j}^{N} A_j)$$

Here A_i and $\cup_{j=1,i\neq j}^{N} A_j$ both are open in $\overline{V}$ and they are disjoint.
$\Rightarrow\ \overline{V}$ having a separation (a contradiction)
Thus our claim is true.
Now we re-order A_i, such that $A_i \cap A_{i+1} \neq \phi, i = 1, 2, ..., N-1$
Let $x, y \in \overline{V}$, the most extreme case is

$$x \in A_1, \quad y \in A_N$$

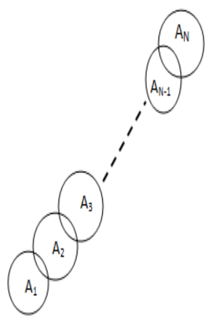
. Then for $z_{12} \in A_1 \cap A_2$

$$|x - z_{12}| \leq r$$

$$\therefore\ u(x) \geq \frac{1}{2^n}u(z_{12})$$

$$z_{23} \in A_2 \cap A_3,\ |z_{12} - z_{23}| \leq r$$

$$u(z_{12}) \geq \frac{1}{2^n}u(z_{23})$$



for

$$z_{34} \in A_3 \cap A_4,\ |z_{23} - z_{34}| \leq r$$

$$u(z_{23}) \geq \frac{1}{2^n}u(z_{34})$$

Similarly

$$z_{N-1,N} \in A_{N-1} \cap A_N$$

$$|z_{(N-1,N)} - y| \leq r$$

$$u(z_{N-1,N}) \geq \frac{1}{2^n}u(y)$$

$$u(x) \geq (\frac{1}{2^n})^N u(y)$$

$$\text{or} \qquad u(x) \geq \frac{1}{2^{nN}} u(y)$$

Similarly

$$u(y) \geq \frac{1}{2^{nN}} u(x)$$

$$\text{thus } \frac{1}{2^{nN}} u(y) \leq u(x) \leq 2^{nN} u(y)$$

$$\frac{1}{C} y(y) \leq u(x) \leq C u(y)$$

Therefore for every $x, y \in V$ $\forall$ a positive constant C such that

$$\frac{1}{C} y(y) \leq u(x) \leq C u(y)$$

Green's Function:

Let us now assume $U \subseteq \mathbb{R}^n$ is open, bounded and ∂U is C^1. Now we will show a systematic way to construct solutions of the boundary value problem for the Poisson's equation

$$- \triangle u = f \text{ in } U, \qquad (1a)$$

subject to the prescribed boundary condition

$$u = g \text{ on } \partial U, \qquad (1b)$$

when the domain U is sufficiently simple (a ball, half space, etc.).
We will construct a more or less explicit formula for the solution. When U is complicated we cannot get as explicit formula but we will reduce solving equation (1) with arbitrary function f and g to the special case $f = 0$, and one particular function g. Having a solution to this one special case allow to construct solutions for general f and g immediately. This is useful when one needs to solve Poisson's equation in the same domain for various f and g .
Let us recall that fundamental solution of the Laplace equation $'\phi(x)'$, which we have obtained earlier is:

$$\phi(x) = -\frac{1}{2\pi} \log |x| \qquad (n = 2)$$

and

$$\phi(x) = \frac{1}{n(n-2)\alpha(n)} \frac{1}{|x|^{(n-2)}} \qquad (n \geq 3).$$

We have shown that

$$u(x) = \int_{\mathbb{R}^n} \phi(x-y) f(y) dy \qquad (2)$$

is a solution of the Poisson's equation

$$- \triangle u = f,$$

posed in all of $\mathbb{R}^n$. Now we would like to adapt the representation formula given by equation (2) to the boundary value problem equation (1a - b) posed in a bounded domain and taking into account the correct boundary conditions.

Derivation of Green's Function:

Suppose first of all $u \in C^2(\overline{U})$ is an arbitrary function. Fix $x \in U$ and choose $\epsilon > 0$ so small that $B(x,\epsilon) \subseteq U$
Now consider the domain

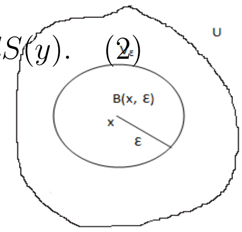
$$V_\epsilon = U - B(x,\epsilon) \quad \text{[i.e. } U \text{ without the ball } B(x,\epsilon)]$$

Now using the Green's formula, we get

$$\int_{V_\epsilon} [u(y) \triangle \phi(y-x) - \phi(y-x) \triangle u(y)]dy = \int_{\partial V_\epsilon} [u(y)\frac{\partial \phi}{\partial \nu}(y-x) - \phi(y-x)\frac{\partial u}{\partial \nu}(y)]dS(y). \tag{1}$$

Here ν denotes the outer unit normal vector on ∂V_ϵ. The reason we had to cut out the small ball around the point x is that now when $y \in V_\epsilon$ the argument $(y-x)$ of the fundamental solution $\phi(y-x)$ cannot vanish, and is regular. Otherwise, we would not be able to apply Green's formula since $\phi(y-x)$ will be singular at $y = x$. As $\triangle\phi(y-x) = 0$ when $y \neq x$, the above relation (1) becomes

$$-\int_{V_\epsilon} \phi(y-x) \triangle u(y)dy = \int_{\partial V_\epsilon} [u(y)\frac{\partial \phi}{\partial \nu}(y-x) - \phi(y-x)\frac{\partial u}{\partial \nu}(y)]dS(y). \tag{2}$$



This identity holds for all $\epsilon > 0$ and we will now pass to the limit $\epsilon \to 0$ in (2). The boundary ∂V_ϵ is the union of ∂U and $\partial B(x,\epsilon)$ as earlier.

$$\begin{aligned}
-\int_{V_\epsilon} \phi(y-x) \triangle u(y)dy &= \int_{\partial U} [u(y)\frac{\partial \phi}{\partial \nu}(y-x) - \phi(y-x)\frac{\partial u}{\partial \nu}(y)]dS(y) \\
&\quad + \int_{\partial B(x,\epsilon)} [u(y)\frac{\partial \phi}{\partial \nu}(y-x) - \phi(y-x)\frac{\partial u}{\partial \nu}(y)]dS(y) \qquad (3)
\end{aligned}$$

Here we will consider the case $n \geq 3$. The case $n = 2$ can be similarly proved. Now

$$\begin{aligned}
\left|\int_{\partial B(x,\epsilon)} \phi(y-x)\frac{\partial u}{\partial \nu}(y)dS(y)\right| &= \frac{1}{n(n-2)\alpha(n)\epsilon^{n-2}} \left|\int_{\partial B(x,\epsilon)} \frac{\partial u}{\partial \nu}(y)dS(y)\right| \\
&\leq \frac{1}{n(n-2)\alpha(n)\epsilon^{n-2}} \int_{\partial B(x,\epsilon)} \left|\frac{\partial u}{\partial \nu}(y)dS(y)\right| \\
&\leq \frac{1}{n(n-2)\alpha(n)\epsilon^{n-2}} M \int_{\partial B(x,\epsilon)} dS(y) \\
&\leq \frac{1}{n(n-2)\alpha(n)\epsilon^{n-2}} Mn\alpha(n)\epsilon^{n-1} \\
&= \frac{M\epsilon}{n-2} \to 0 \text{ as } \epsilon \to 0
\end{aligned}$$

here $M = \sup_{y\in U} |Du|$.
For $n \geq 3$,

$$|y-x|^2 = \left[(y_1-x_1)^2 + (y_2-x_2)^2 + (y_3-x_3)^2 + \ \ldots \ + (y_n-x_n)^2\right]$$

$$\begin{aligned}
\frac{\partial\phi(y-x)}{\partial\nu} &= D\phi(y-x).\nu = D\left[\frac{1}{n(n-2)\alpha(n)|y-x|^{n-2}}\right].\bar{\nu} \quad \left(\text{where, } \bar{\nu} = \frac{-(y-x)}{|y-x|}\right) \\
&= \left[\frac{-(n-2)|y-x|^{(-n+2-1)}}{n(n-2)\alpha(n)}\sum_{i=1}^{n}\frac{\partial(|y-x|)}{\partial y_i}\right]\cdot\left[\frac{-(y-x)}{|y-x|}\right] \\
&= \left[\frac{-(n-2)|y-x|^{-n}(y-x)}{n(n-2)\alpha(n)}\right]\cdot\left[\frac{-(y-x)}{|y-x|}\right] \\
&= \frac{(y-x)\cdot(y-x)}{n\alpha(n)|y-x|^{n+1}} = \frac{|y-x|^2}{n\alpha(n)|y-x|^{n+1}} \\
&= \frac{1}{n\alpha(n)|y-x|^{n-1}} = \frac{1}{n\alpha(n)\epsilon^{n-1}} \quad \text{On } \ \partial B(x,\epsilon)
\end{aligned}$$

$$\begin{aligned}
\int_{\partial B(x,\epsilon)} u(y)\frac{\partial\phi}{\partial\nu}(y-x)dS(y) &= \frac{1}{n\alpha(n)\epsilon^{n-1}}\int_{\partial B(x,\epsilon)} u(y)dS(y) \\
&= \frac{1}{|\partial B(x,\epsilon)|}\int_{\partial B(x,\epsilon)} u(y)dS(y) \\
&= \fint_{\partial B(x,\epsilon)} u(y)dS(y) \to u(x)\text{as } \epsilon \to 0.
\end{aligned}$$

Now sending $\epsilon \to 0$ in (3), we get

$$-\int_U \phi(y-x)\triangle u(y)dy = \int_{\partial U}[u(y)\frac{\partial\phi}{\partial\nu}(y-x) - \phi(y-x)\frac{\partial u}{\partial\nu}(y)]dS(y) + u(x)$$

$$u(x) = \int_{\partial U}[\phi(y-x)\frac{\partial u}{\partial\nu}(y) - u(y)\frac{\partial\phi}{\partial\nu}(y-x)]dS(y) - \int_U \phi(y-x)\triangle u(y)dy. \quad (3)$$

This identity is valid for any point $x \in U$ and any function $u \in C^2(\overline{U})$.
Therefore in order to compute $u(x)$ we should know $\triangle u$ insides U(which we do for the solution of the Poisson's equation (1a), it is f), as well as $u(y)$ on the boundary ∂U(which we do know for the solution of the boundary value problem (1b), it is g), but also the normal derivative $\frac{\partial u}{\partial\nu}$ at the boundary of U and that we do not know.
We must therefore some how modify to remove this term.
The idea is now to introduce for fixed x a corrector function $\phi^x = \phi^x(y)$, solving the boundary value problem:

$$\triangle\phi^x = 0 \quad \text{in } \ U$$

$$\phi^x = \phi(y-x) \quad \text{on } \ \partial U$$

Let us apply Green's formula once more (but without the need to throw out a small ball around the point x, since the function ϕ^x is regular at $y = x$) gives

$$\begin{aligned} -\int_U \phi^x(y) \triangle u(y)dy &= \int_{\partial U} [u(y)\frac{\partial \phi^x}{\partial \nu}(y) - \phi^x(y)\frac{\partial u}{\partial \nu}(y)]dS(y) \\ &= \int_{\partial U} u(y)\frac{\partial \phi^x}{\partial \nu}(y)dS(y) - \int_{\partial U} \phi(y-x)\frac{\partial u}{\partial \nu}(y)dS(y) \end{aligned}$$

$$\int_{\partial U} \phi(y-x)\frac{\partial u}{\partial \nu}(y)dS(y) = \int_{\partial U} u(y)\frac{\partial \phi^x}{\partial \nu}(y)dS(y) + \int_U \phi^x(y) \triangle u(y)dy.$$

Using this in (4), we get

$$\begin{aligned} u(x) &= \int_{\partial U} u(y)\frac{\partial \phi^x}{\partial \nu}(y)dS(y) + \int_U \phi^x(y) \triangle u(y)dy - \int_{\partial U} u(y)\frac{\partial \phi}{\partial \nu}(y-x)dS(y) \\ &- \int_U \phi(y-x) \triangle u(y)dy \end{aligned}$$

$$u(x) = -\int_{\partial U} u(y)[\frac{\partial \phi}{\partial \nu}(y-x) - \frac{\partial \phi^x}{\partial \nu}(y)]dS(y) - \int_U [\phi(y-x) - \phi^x(y)] \triangle u(y)dy. \quad (5)$$

<u>Definition:</u> Green's function for the region U is

$$G(x,y) = \phi(y-x) - \phi^x(y) \quad (x, y \in U, x \neq y).$$

Adapting this terminology in (5), we get

$$u(x) = -\int_{\partial U} u(y)\frac{\partial G}{\partial \nu}(x,y)dS(y) - \int_U G(x,y) \triangle u(y)dy \quad (x \in U) \qquad (6)$$

where

$$\frac{\partial G}{\partial \nu}(x,y) = D_y G(x,y) \cdot \nu(y)$$

is the outer normal derivative of G with respect to the variable y. It is easy to note that the term $\frac{\partial u}{\partial \nu}$ does not appear in (6). We have introduced the corrector function ϕ^x precisely to achieve this.

<u>Theorem:</u> (Representation formula using Green's function).
Suppose now $u \in C^2(\overline{U})$ solves the boundary value problem

$$-\triangle u = f \quad \text{in} \quad U \qquad (7)$$

$$u = g \quad \text{on} \quad \partial U$$

for given continuous functions f and g. Put this into (6), we obtain the following result.

$$u(x) = -\int_{\partial U} g(y)\frac{\partial G}{\partial \nu}(x,y)dS(y) + \int_U f(y)G(x,y)dy \quad (x \in U).$$

Here we have a formula for the solution of the boundary value problem (7) provided we can construct Green's function G for the given domain U. This is in general a difficult matter and can be done only when U has simple geometry.

<u>Theorem:</u> (Symmetry of Green's function)
For all $x, y \in U, \quad x \neq y$, we have

$$G(y,x) = G(x,y)$$

where Green's function is

$$G(x,y) = \phi(y-x) - \phi^x(y)$$

Proof: Now to prove above theorem, using Green's formula. Let $x \neq y$ be two distinct points in U, and set $\nu(z) = G(x,z)$ and $w(z) = G(y,z)$. Let us at two small balls $B(x,\epsilon)$ and $B(y,\epsilon)$ with $\epsilon > 0$ so small that the balls are not overlapping and are contained in U. Let $V = U - [B(x,\epsilon) \bigcup B(y,\epsilon)]$ be the domain U with the two balls deleted.

$$\left.\begin{array}{llll} \text{Then} \quad \triangle\, v(z) = 0 & z \neq x & z \in U \\ \triangle w(z) = 0 & z \neq y & z \in U \\ \text{and} \quad v = w = 0 & \text{on} & \partial U \end{array}\right\} \qquad (1)$$

and $B(x,\epsilon) \subseteq U \ \ \& \ \ B(y,\epsilon) \subseteq U$.
i.e. $\triangle_z w = \triangle_z v = 0$ in V as this set contains neither the point x nor the point y. The Green's formula then becomes

$$\int_{\partial V} \left[w(z) \frac{\partial v}{\partial \nu} - v(z) \frac{\partial w(z)}{\partial \nu} \right] dS(z) = \int_V [w(z) \triangle v(z) - v(z) \triangle w(z)] dz = 0 \text{ (by (1))} \quad (2)$$

The boundary of V consists of three pieces, the outer boundary ∂U where both w and v vanish and the two spheres $\partial B(x,\epsilon)$ and $\partial B(y,\epsilon)$

$$\text{i.e. } \partial V = \partial U \bigcup \partial B(x,\epsilon) \bigcup \partial B(y,\epsilon)$$

$$\text{and } v = w = 0 \ \ on \ \ \partial U.$$

Therefore from (2), we have

$$\int_{\partial B(x,\epsilon)} \left[w(z) \frac{\partial v(z)}{\partial \nu} - v(z) \frac{\partial w(z)}{\partial \nu} \right] dS(z) + \int_{\partial B(y,\epsilon)} \left[w(z) \frac{\partial v(z)}{\partial \nu} - v(z) \frac{\partial w(z)}{\partial \nu} \right] dS(z)$$

$$+ \int_{\partial U} \left[w(z) \frac{\partial v(z)}{\partial \nu} - v(z) \frac{\partial w(z)}{\partial \nu} \right] dS(z) = 0 \qquad (3)$$

$$\because w(z) = G(y,z) = \phi(z-y) - \phi^y(z)$$

$$\phi^y(z) = \phi(z-y) \text{ on } \partial U$$

$$w(z) = \phi(z-y) - \phi(z-y) = 0 \text{ on } \partial U$$

$$w(z) = 0 \text{ on } \partial U$$

$$\text{similarly } \ v(z) = 0 \text{ on } \partial U$$

Now from (3),

$$\int_{\partial B(x,\epsilon)} \left[w(z)\frac{\partial v(z)}{\partial \nu} - v(z)\frac{\partial w(z)}{\partial \nu} \right] dS(z) + \int_{\partial B(y,\epsilon)} \left[w(z)\frac{\partial v(z)}{\partial \nu} - v(z)\frac{\partial w(z)}{\partial \nu} \right] dS(z) = 0 \qquad (4)$$

$v(z)$ is harmonic in $B(y,\epsilon)$ and $w(z)$ is harmonic in $B(x,\epsilon)$ where ν denotes the inward pointing unitnormal on $\partial B(x,\epsilon) \bigcup \partial B(y,\epsilon)$.
Now since w is smooth except for $z = y$, therefore $\frac{\partial w}{\partial \nu}$ is bounded on $\partial B(x,\epsilon)$.
So $\exists\ M > 0$ such that $|\frac{\partial w}{\partial \nu}| \leq M$ on $\partial B(x,\epsilon)$

$$\begin{aligned}
\text{Now } \left| \int_{\partial B(x,\epsilon)} v(z)\frac{\partial w(z)}{\partial \nu} dS(z) \right| &\leq \int_{\partial B(x,\epsilon)} |v(z)| \left| \frac{\partial w(z)}{\partial \nu} \right| dS(z) \\
&\leq \sup_{z \in \partial B(x,\epsilon)} |v(z)| M \int_{\partial B(x,\epsilon)} dS(z) \\
&\leq \sup_{z \in \partial B(x,\epsilon)} |v(z)| M n\alpha(n)\epsilon^{n-1} \longrightarrow 0 \text{ as } \epsilon \longrightarrow 0 \\
\lim_{\epsilon \longrightarrow 0} \int_{\partial B(x,\epsilon)} v(z)\frac{\partial w(z)}{\partial \nu} dS(z) &= 0
\end{aligned}$$

Similarly, we have

$$\lim_{\epsilon \longrightarrow 0} \int_{\partial B(y,\epsilon)} w(z)\frac{\partial v(z)}{\partial \nu} dS(z) = 0$$

Therefore, taking $\epsilon \longrightarrow 0$ in equation (4), we have

$$\lim_{\epsilon \longrightarrow 0} \left\{ \int_{\partial B(x,\epsilon)} w(z)\frac{\partial v(z)}{\partial \nu} dS(z) - \int_{\partial B(y,\epsilon)} v(z)\frac{\partial w(z)}{\partial \nu} dS(z) \right\} = 0$$

$$\lim_{\epsilon \longrightarrow 0} \int_{\partial B(x,\epsilon)} w(z)\frac{\partial v(z)}{\partial \nu} dS(z) = \lim_{\epsilon \longrightarrow 0} \int_{\partial B(y,\epsilon)} v(z)\frac{\partial w(z)}{\partial \nu} dS(z) \qquad (5)$$

Since $v(z) = \phi(z-x) - \phi^x(z)$ in U and $\phi^x(z)$ is smooth in U

$$\begin{aligned}
\lim_{\epsilon \longrightarrow 0} \int_{\partial B(x,\epsilon)} w(z)\frac{\partial v(z)}{\partial \nu} dS(z) &= \lim_{\epsilon \longrightarrow 0} \int_{\partial B(x,\epsilon)} w(z)\frac{\partial \phi(z-x)}{\partial \nu} dS(z) \\
&\quad - \lim_{\epsilon \longrightarrow 0} \int_{\partial B(x,\epsilon)} w(z)\frac{\partial \phi^x(z)}{\partial \nu} dS(z)
\end{aligned}$$

Claim: $\lim_{\epsilon \longrightarrow 0} \int_{\partial B(x,\epsilon)} w(z)\frac{\partial \phi^x(z)}{\partial \nu} dS(z) = 0$?
As $v(z)$ is harmonic in $B(y,\epsilon)$ and $w(z)$ is harmonic in $B(x,\epsilon)$

$$v(z) = G(x,z) = \phi(z-x) - \phi^x(z)$$

$$\triangle\phi^x = 0 \text{ in } U \text{ and so in } B(x,\epsilon)$$

$$\phi^x(z) = \phi(z-x) \;\text{ on Boundary } \partial U$$

we know the Green's formula

$$\int_U (Du.Dv)dx = -\int_U u \triangle v dx + \int_{\partial U} u\frac{\partial v}{\partial \nu} dS$$

Now taking $v = \phi^x(z)$ and $u = w(z)$, the above formula becomes

$$\int_{B(x,\epsilon)} (Dw(z)\cdot D\phi^x(z))dz = -\int_{B(x,\epsilon)} w(z) \triangle \phi^x(z)dz + \int_{\partial B(x,\epsilon)} w(z)\frac{\partial \phi^x(z)}{\partial \nu} dS(z)$$

Also $\triangle\phi^x(z) = 0$ in $B(x,\epsilon)$ and so

$$\int_{\partial B(x,\epsilon)} w(z)\frac{\partial \phi^x(z)}{\partial \nu} dS(z) = \int_{B(x,\epsilon)} (Dw(z)\cdot D\phi^x(z))dz$$

Since,

$$\left|\int_{\partial B(x,\epsilon)} w(z)\frac{\partial \phi^x(z)}{\partial \nu} dS(z)\right| \leq C\epsilon^{(n-1)} \sup_{\partial B(x,\epsilon)} |Dw| \longrightarrow 0 \text{ as } \epsilon \longrightarrow 0.$$

$$\begin{aligned}
\lim_{\epsilon\longrightarrow 0}\int_{\partial B(x,\epsilon)} w(z)\frac{\partial \nu(z)}{\partial \nu} dS(z) &= \lim_{\epsilon\longrightarrow 0}\int_{\partial B(x,\epsilon)} w(z)\frac{\partial \phi(z-x)}{\partial \nu} dS(z) - 0\\
&= \lim_{\epsilon\longrightarrow 0} \frac{1}{n\alpha(n)\epsilon^{n-1}}\int_{\partial B(x,\epsilon)} w(z)dS(z)\\
&= \lim_{\epsilon\longrightarrow 0} \frac{1}{|\partial B(x.\epsilon)|}\int_{\partial B(x,\epsilon)} w(z)dS(z)\\
&= \lim_{\epsilon\longrightarrow 0} \oint_{\partial B(x,\epsilon)} w(z)dS(z)\\
&= w(x) = G(y,x).
\end{aligned}$$

Similarly, we can show that

$$\lim_{\epsilon\longrightarrow 0}\int_{\partial B(y,\epsilon)} v(z)\frac{\partial w(z)}{\partial \nu} dS(z) = v(y) = G(x,y)$$

Now from equation (5), we have

$$\boxed{G(y,x) = G(x,y)}$$

This proves symmetry of Green's functions.

Green function for half space:

The half space is defined as

$$\mathbb{R}^n_+ = \{x \;=\; (x_1, x_2, x_3, \;.\;.\;.\;, x_n) \in \mathbb{R}^n : x_n > 0\}$$

This region is unbounded.

Definition: If $x = (x_1, x_2, x_3, \ . \ . \ . \ , x_n) \in \mathbb{R}^n$ then its reflection in the plane $\partial\mathbb{R}^n_+$ is the point

$$\tilde{x} = (x_1, x_2, x_3, \ . \ . \ . \ , -x_n) \in \mathbb{R}^n$$

$$\partial\mathbb{R}^n_+ = \{(x_1, x_2, x_3, \ . \ . \ . \ , x_n); \ x_n = 0\}$$

We will solve the problem

$$\left.\begin{aligned} \triangle\phi^x &= 0 \ \text{ in } \mathbb{R}^n_+ \\ \phi^x &= \phi(y-x) \ \text{ on } \ \partial\mathbb{R}^n_+ \end{aligned}\right\} \qquad (1)$$

Let

$$\phi^x(y) = \phi(y-\tilde{x}) = \phi(y_1 - x_1, y_2 - x_2, y_3 - x_3, \ . \ . \ . \ y_{n-1} - x_{n-1}, y_n + x_n) \ \ x \in \mathbb{R}^n_+, \ \tilde{x} \notin \mathbb{R}^n_+$$

we note that ϕ^x is analytic in the region $\mathbb{R}^n_+$ and $\triangle\phi^x = 0$ in $\mathbb{R}^n_+$ and $\phi^x(y) = \phi(y-x)$ on $\partial\mathbb{R}^n_+$.
Thus, $\phi^x(y)$ satisfies the condition of correcter function.
Now Green's function for the half - space is

$$G(x,y) = \phi(y-x) - \phi(y-\tilde{x}), \quad x, y \in \mathbb{R}^n_+ \ \text{ and } \ x \neq y.$$

For the half - space, we will calculate

$$\frac{\partial G(x,y)}{\partial y_n} = \frac{\partial\phi(y-x)}{\partial y_n} - \frac{\partial\phi(y-\tilde{x})}{\partial y_n} \qquad (1)$$

Now, $\frac{\partial\phi(y-x)}{\partial y_n}$

$$\because \ \phi(y-x) = \frac{1}{n(n-2)\alpha(n)} \frac{1}{r^{(n-2)}}, \ n \geq 3 \ \text{ where } r = |y-x|$$

$$\text{where } \ r^2 = (y_1 - x_1)^2 + (y_2 - x_2)^2 + (y_3 - x_3)^2 + \ . \ . \ . \ + (y_n - x_n)^2$$

$$2r\frac{\partial r}{\partial y_n} = 2(y_n - x_n)$$

$$\Longrightarrow \ \frac{\partial r}{\partial y_n} = \frac{(y_n - x_n)}{r}$$

$$\begin{aligned} \frac{\partial\phi(y-x)}{\partial y_n} &= \frac{\partial}{\partial y_n}\left(\frac{1}{n(n-2)\alpha(n)}\frac{1}{r^{n-2}}\right) \\ &= \frac{1}{n(n-2)\alpha(n)}\frac{\partial}{\partial y_n}(r^{-n+2}) \\ &= \frac{1}{n(n-2)\alpha(n)}((2-n)r^{-n+1})\frac{\partial r}{\partial y_n} \\ &= \frac{1}{n(n-2)\alpha(n)}\left(\frac{-(n-2)}{r^{n-1}}\right)\frac{(y_n - x_n)}{r} \end{aligned}$$

$$\begin{aligned}
\frac{\partial\phi(y-x)}{\partial y_n} &= -\frac{1}{n\alpha(n)}\frac{y_n-x_n}{r^n}, \quad r=|y-x| \quad (2)\\
\text{Now,} \qquad \frac{\partial\phi(y-\tilde{x})}{\partial y_n} &= \frac{\partial}{\partial y_n}\left(\frac{1}{n\alpha(n)(n-2)}\frac{1}{|y-\tilde{x}|^{n-2}}\right)\\
|y-\tilde{x}| &= \left((y_1-x_1)^2+(y_2-x_2)^2+\ \ldots\ +(y_n+x_n)^2\right)^{\frac{1}{2}}\\
\frac{\partial}{\partial y_n}(|y-\tilde{x}|) &= \frac{y_n+x_n}{|y-\tilde{x}|}\\
\text{Thus,} \qquad \frac{\partial\phi^x(y)}{\partial y_n} &= \frac{-(n-2)}{n(n-2)\alpha(n)}\frac{1}{|y-\tilde{x}|^{n-1}}\frac{\partial}{\partial y_n}(|y-\tilde{x}|)\\
\frac{\partial\phi(y-\tilde{x})}{\partial y_n} &= \frac{-1}{n\alpha(n)}\frac{y_n+x_n}{|y-\tilde{x}|^n} \qquad (3)
\end{aligned}$$

Now putting the values of (2) and (3) in (1), we get

$$\frac{\partial G(x,y)}{\partial y_n} = \frac{-1}{n\alpha(n)}\left[\frac{y_n-x_n}{|y-x|^n}-\frac{y_n+x_n}{|y-\tilde{x}|^n}\right] = \frac{2x_n}{n\alpha(n)|y-x|^n}$$

Consequently if $y\in\partial\mathbb{R}^n_+$, then

$$\frac{\partial G(x,y)}{\partial\nu} = -\frac{\partial G(x,y)}{\partial y_n} = -\frac{2x_n}{n\alpha(n)|y-x|^n}.$$

Now suppose u solves the Boundary value problem

$$\triangle u = 0 \ \text{ in } \mathbb{R}^n_+$$

$$u = g \ \text{ on } \ \partial\mathbb{R}^n_+$$

Then by representation formula using Green's function, we have

$$\begin{aligned}
u(x) &= -\int_{\mathbb{R}^n_+} G(x,y)\triangle u(y)dy - \int_{\partial\mathbb{R}^n_+} u(y)\frac{\partial G(x,y)}{\partial\nu}dS(y)\\
&= 0 - \int_{\partial\mathbb{R}^n_+} g(y)\left\{\frac{-2x_n}{n\alpha(n)|y-x|^n}\right\}dS(y)\\
u(x) &= \frac{2x_n}{n\alpha(n)}\int_{\partial\mathbb{R}^n_+}\frac{g(y)}{|x-y|^n}dS(y)\\
u(x) &= \int_{\partial\mathbb{R}^n_+} K(x,y)g(y)dy_1dy_2\ \ldots\ dy_{n-1} \quad (4)
\end{aligned}$$

where, $K(x,y)=\frac{2x_n}{n\alpha(n)}\frac{1}{|x-y|^n}$, $x\in\mathbb{R}^n_+$, $y\in\mathbb{R}^n_+$ is known as Poisson's Kernal.
(4) is the representation formula for our solution

$$\because\ u(x) = -\int_{\partial U} g\frac{\partial G}{\partial\nu}dS(y) + \int_U fG(x,y)dy \quad (\because -\triangle u = 0 = f)$$

Theorem: Assume $g \in C(\mathbb{R}^{n-1}) \bigcap L^{\infty}(\mathbb{R}^{n-1})$ and define u by $u(x) = \int_{\partial\mathbb{R}^n_+} K(x,y)g(y)dy$. Then

$$(i)\ u \in C^{\infty}(\mathbb{R}^n_+) \bigcap L^{\infty}(\mathbb{R}^n_+)$$
$$(ii)\ \triangle\, u = 0 \ \text{ in } \mathbb{R}^n_+$$
$$(iii)\ \lim_{x \to x_0, x \in \mathbb{R}^n_+} u(x) = g(x_0) \ \text{ for each point } \ x_0 \in \partial\mathbb{R}^n_+$$

Proof: (i) For each point x, the mapping $x \longrightarrow G(x,y)$ is harmonic except for $y = x$.
As $G(x,y) = G(y,x)$ then $x \longrightarrow G(x,y)$ is harmonic except for $y = x$.
Thus $x \longrightarrow -\frac{\partial G(x,y)}{\partial y_n} = K(x,y)$ is harmonic for $x \in \mathbb{R}^n_+,\ y \in \partial\mathbb{R}^n_+$.
So keeping in mind harmonicity of function $K(x,y)$, (i) is overed.
(ii) A direct calculation yields that

$$\int_{\partial\mathbb{R}^n_+} K(x,y)dy = 1, \quad \text{for each} \quad x \in \mathbb{R}^n_+.$$

As g is bounded, u defined above like wise bounded. Since $x \longrightarrow K(x,y)$ is smooth for $x \neq y$, we can easily say that $u \in C^{\infty}(\mathbb{R}^n_+)$

$$\text{and} \quad \triangle\, u \ = \ \int_{\partial\mathbb{R}^n_+} \triangle_x(K(x,y))g(y)dy, \qquad x \in \mathbb{R}^n_+$$
$$\text{this implies that} \quad \triangle\, u \ = \ 0 \quad \text{in} \quad \mathbb{R}^n_+$$

(iii) We have

$$\begin{aligned}
\triangle u &= 0 \ \text{ in } \mathbb{R}^n_+ \\
u &= g \quad \text{on} \ \ \partial\mathbb{R}^n_+ \\
u(x) &= \frac{2x_n}{n\alpha(n)} \int_{\partial\mathbb{R}^n_+} \frac{g(y)dy}{|x-y|^n}, \quad x \in \mathbb{R}^n_+ \qquad (1) \\
K(x,y) &= \frac{2x_n}{n\alpha(n)} \frac{1}{|x-y|^n} \quad (x \in \mathbb{R}^n_+, y \in \partial\mathbb{R}^n_+).
\end{aligned}$$

Now for fixed $x_0 \in \partial\mathbb{R}^n_+,\ \epsilon > 0,$ choose $\delta > 0$ so small that

$$|g(y) - g(x_0)| < \epsilon \ \text{ if } \ |y - x_0| < \delta,\ y \in \partial\mathbb{R}^n_+$$

Now for $|x - x_0| < \frac{\delta}{2},\ x \in \mathbb{R}^n_+,$ we have

$$\begin{aligned}
|u(x) - g(x_0)| &= \left| \int_{\partial\mathbb{R}^n_+} K(x,y)(g(y) - g(x_0))dy \right| \\
&\leq \int_{\partial\mathbb{R}^n_+} K(x,y)|(g(y) - g(x_0))|dy \\
&\leq \int_{\partial\mathbb{R}^n_+ \bigcap B(x_0,\delta)} K(x,y)|g(y) - g(x_0)|dy + \int_{\partial\mathbb{R}^n_+ - B(x_0,\delta)} K(x,y)|g(y) - g(x_0)|dy \\
&\quad (\because \int_{\partial\mathbb{R}^n_+} K(x,y)dy = 1) \\
&\leq \int_{\partial\mathbb{R}^n_+ \bigcap B(x_0,\delta)} K(x,y)|g(y) - g(x_0)|dy + \int_{\partial\mathbb{R}^n_+ - B(x_0,\delta)} K(x,y)|g(y) - g(x_0)|dy
\end{aligned}$$

Hence, we note that

$$\begin{aligned}
I &= \int_{\partial\mathbb{R}^n_+ \bigcap B(x_0,\delta)} K(x,y)|g(y)-g(x_0)|dy \\
&< \epsilon \int_{\partial\mathbb{R}^n_+} K(x,y)dy = \epsilon \text{ since } \because \int_{\partial\mathbb{R}^n_+} K(x,y)dy = 1 \\
\text{and } J &= \int_{\partial\mathbb{R}^n_+ - B(x_0,\delta)} |g(y)-g(x_0)|dy.
\end{aligned}$$

For $|x-x_0| < \frac{\delta}{2}$ and $|y-x_0| \geq \delta$, we have

$$\begin{aligned}
|y-x_0| &\leq |y-x| + |x-x_0| \\
|y-x_0| &\leq |y-x| + \frac{\delta}{2} \leq |y-x| + \frac{1}{2}|y-x_0| \\
\text{and so} \quad |y-x| &\geq \frac{1}{2}|y-x_0| \\
\text{Thus,} \quad J &\leq 2\|g\|_{L^\infty} \int_{\partial\mathbb{R}^n_+ - B(x_0,\delta)} K(x,y)dy \\
&\leq \frac{2^{n+2}\|g\|_{L^\infty} x_n}{n\alpha(n)} \int_{\partial\mathbb{R}^n_+ - B(x_0,\delta)} |y-x_0|^{-n} dy \longrightarrow 0, \text{ as } x_n \longrightarrow 0
\end{aligned}$$

Now we deduced that

$$|u(x) - g(x_0)| \leq 2\epsilon, \text{ provided } |x-x_0| \text{ is sufficiently small}$$

This implies that $\lim_{x \longrightarrow x_0, x \in \mathbb{R}^n_+} u(x) = g(x_0)$, for each point $x_0 \in \partial\mathbb{R}^n_+$

Green's function for unit ball:

Definition: If $x \in \mathbb{R}^n - \{0\}$ then the point $\tilde{x} = \frac{x}{|x|^2}$ is called the point dual to x with respect to $\partial B(0,1)$. The mapping $x \longrightarrow \tilde{x}$ is inversion through unit sphere $\partial B(0,1)$.
Now we will take $U = B^0(0,1)$ and for $x \in B^0(0,1)$, we must find a correcter function $\phi^x = \phi^x(y)$ solving

$$\triangle \phi^x = 0 \text{ in } B^0(0,1)$$

$$\phi^x = \phi(y-x) \text{ and } \partial B^0(0,1)$$

Then the Green's function will be

$$G(x,y) = \phi(y-x) - \phi^x(y).$$

The idea now is to invert the singularity from $x \in B^0(0,1)$ to $\tilde{x} \notin B(0,1)$

$$\because x \in B^0(0,1) \Longrightarrow |x-0| < 1 \Longrightarrow |x| < 1$$

$$\because \tilde{x} = \frac{x}{|x|^2} \quad (\because x \in \mathbb{R}^n - \{0\})$$

$$\begin{aligned}\Longrightarrow |\tilde{x}| &= \frac{1}{|x|} \Longrightarrow |\tilde{x}| > 1\\ \Longrightarrow \tilde{x} &\notin B(0,1).\end{aligned}$$

Then the mapping $y \to \phi(y-\tilde{x})$ is harmonic as $y \neq \tilde{x}$.
Thus $y \to |x|^{-(n-2)}\phi(y-\tilde{x})$ is harmonic for $n \geq 3$ and so $\phi^x(y) = \phi(|x|(y-\tilde{x}))$ is harmonic in U. If $y \in \partial B(0,1)$ and $x \neq 0$, then

$$\begin{aligned}|x|^2|y-\tilde{x}|^2 &= |x|^2[|y|^2 + |\tilde{x}|^2 - 2y\tilde{x}]\\ &= |x|^2\left[|y|^2 + \left(\frac{x}{|x|^2}\right)^2 - 2y\frac{x}{|x|^2}\right]\\ &= \left[|x|^2 + \frac{|x|^2}{|x|^2} - 2yx\right]\\ &= \left[|x|^2 - 2yx + 1\right]\\ &= \left[|x|^2 - 2yx + |y|^2\right] \quad (\because\ |y|^2 = 1)\end{aligned}$$

$$\begin{aligned}|x|^2|y-\tilde{x}|^2 &= |x-y|^2\\ \text{so } [|x|(y-\tilde{x})]^{-(n-2)} &= |x-y|^{-(n-2)}\\ \text{Now } \because\ \phi^x(y) &= \phi(|x||(y-\tilde{x})|)\\ &= \frac{1}{n(n-2)\alpha(n)}\frac{1}{(|x||y-\tilde{x}|)^{n-2}}\\ &= \frac{1}{n(n-2)\alpha(n)}\frac{1}{|x-y|^{n-2}}, \quad \text{if } y \in \partial B(0,1)\\ &= \phi(y-x)\\ \phi^x(y) &= \phi(y-x) \quad \text{on } \partial B(0,1).\\ \text{Now } G(x,y) &= \phi(y-x) - \phi(|x||y-\tilde{x}|),\ x \in B(0,1),\ y \in \partial B(0,1),\ x \neq y\\ G(x,y) &= \phi(y-x) - \phi(y-x)\\ G(x,y) &= 0 \quad \text{when } x \in B(0,1)\ \&\ y \in \partial B(0,1).\end{aligned}$$

Assume u solves the boundary value problem

$$\begin{aligned}\triangle u &= 0 \text{ in } B(0,1)\\ u &= g \text{ on } \partial B(0,1)\end{aligned}$$

$\because$ $B^0(0,1)$ is open, bounded and $\partial B(0,1)$ in C^1. Now the solution of above problem is

$$u(x) = -\int_{\partial B(0,1)} g(y)\frac{\partial G(x,y)}{\partial \nu}dS(y) \qquad (1)$$

Now we will find $\frac{\partial G(x,y)}{\partial \nu}$

$$
\begin{aligned}
\because \ \frac{\partial G(x,y)}{\partial \nu} &= D_y G(x,y).\nu(y) \\
\text{Now} \ \frac{\partial G(x,y)}{\partial y_i} &= \frac{\partial \phi(y-x)}{\partial y_i} - \frac{\partial}{\partial y_i}\phi(|x||y-\tilde{x}|) \qquad (2) \\
\because \ \frac{\partial \phi(y-x)}{\partial y_i} &= \frac{\partial}{\partial y_i}\left\{\frac{1}{n(n-2)\alpha(n)}\frac{1}{|y-x|^{n-2}}\right\} \\
&= -\frac{1}{n(n-2)\alpha(n)}(n-2)|y-x|^{1-n}\frac{\partial}{\partial y_i}(|y-x|) \\
&= -\frac{1}{n\alpha(n)|y-x|^{n-1}}\frac{\partial}{\partial y_i}\{(y_1-x_1)^2+(y_2-x_2)^2+\ \ldots\ +(y_n-x_n)^2\}^{\frac{1}{2}} \\
&= -\frac{1}{n\alpha(n)|y-x|^{n-1}}\frac{1}{2}\{(y_1-x_1)^2+(y_2-x_2)^2+\ \ldots\ +(y_n-x_n)^2\}^{-\frac{1}{2}}2(y_i-x_i) \\
&= -\frac{1}{n\alpha(n)|y-x|^{n-1}}\frac{y_i-x_i}{|y-x|} \\
&= -\frac{1}{n\alpha(n)}\frac{(y_i-x_i)}{|y-x|^n} \\
\text{or} \ \frac{\partial \phi(y-x)}{\partial y_i} &= \frac{1}{n\alpha(n)}\frac{(x_i-y_i)}{|x-y|^n} \qquad (3) \\
\frac{\partial \phi(|y-\tilde{x}||x|)}{\partial y_i} &= \frac{\partial}{\partial y_i}\left\{\frac{1}{n(n-2)\alpha(n)}\frac{1}{|x|^{n-2}|y-\tilde{x}|^{n-2}}\right\} \\
&= \frac{1}{n(n-2)\alpha(n)}\frac{(1)}{|x|^{n-2}}\frac{\partial}{\partial y_i}(|y-\tilde{x}|^{-n+2}) \\
&= \frac{1}{n(n-2)\alpha(n)}\frac{(1)}{|x|^{n-2}}(-n+2)(|y-\tilde{x}|^{-n+1})\frac{\partial}{\partial y_i}(|y-\tilde{x}|) \\
&= -\frac{1}{n\alpha(n)|x|^{n-2}}\frac{\partial}{\partial y_i}\left((y_1-\tilde{x_1})^2+\ \ldots\ +(y_n-\tilde{x_n})^2\right)^{\frac{1}{2}} \\
&= -\frac{1}{n\alpha(n)|x|^{n-2}}\frac{1}{|y-\tilde{x}|^{n-1}}\left((y_1-\tilde{x_1})^2+\ \ldots\ +(y_n-\tilde{x_n})^2\right)^{\frac{-1}{2}}(y_i-\tilde{x_i}) \\
&= -\frac{1}{n\alpha(n)|x|^{n-2}}\frac{1}{|y-\tilde{x}|^{n-1}}\frac{1}{|y-\tilde{x}|}(y_i-\tilde{x_i}) \\
&= -\frac{1}{n\alpha(n)|x|^{n-2}}\frac{1}{|y-\tilde{x}|^{n}}(y_i-\frac{x_i}{|x|^2}) \\
&= -\frac{1}{n\alpha(n)|x|^{n-2}}\frac{1}{|y-\tilde{x}|^{n}}\left(\frac{y_i|x|^2-x_i}{|x|^2}\right) \\
&= -\frac{1}{n\alpha(n)}\left(\frac{y_i|x|^2-x_i}{|x|^n|y-\tilde{x}|^n}\right) \\
\Longrightarrow \ \frac{\partial \phi(|y-\tilde{x}||x|)}{\partial y_i} &= -\frac{1}{n\alpha(n)}\left(\frac{y_i|x|^2-x_i}{|x-y|^n}\right) \\
\text{and} \quad \nu(y) &= \frac{y}{|y|^2} = y \ \ (\because\ |y|=1,\ y\in\partial B(0,1))
\end{aligned}
$$

$$\text{Now,}\quad \begin{aligned}\frac{\partial G(x,y)}{\partial \nu} &= \sum_{i=1}^{n} y_i \frac{\partial G}{\partial y_i}(x,y)\\
&= \sum_{i=1}^{n} y_i \left\{ -\frac{1}{n\alpha(n)} \frac{y_i - x_i}{|y-x|^n} - \left(-\frac{1}{n\alpha(n)} \frac{(y_i|x|^2 - x_i)}{|x-y|^n} \right) \right\}\\
&= \frac{-1}{n\alpha(n)|x-y|^n} \sum_{i=1}^{n} y_i \left\{ y_i - y_i|x|^n \right\}\\
&= \frac{-1}{n\alpha(n)|x-y|^n} \sum_{i=1}^{n} y_i^2 \left(1 - |x|^2\right)\\
&= \frac{-1}{n\alpha(n)|x-y|^n} \left(1 - |x|^2\right) \sum_{i=1}^{n} y_i^2\\
&= \frac{-1}{n\alpha(n)} \frac{(1-|x|^2)}{|x-y|^n}\end{aligned}$$

Hence, from equation (1), we have

$$\begin{aligned} u(x) &= \int_{\partial B(0,1)} \frac{(1-|x|^2)}{n\alpha(n)} \frac{g(y)}{|x-y|^n} dS(y)\\
&= \frac{1-|x|^2}{n\alpha(n)} \int_{\partial B(0,1)} \frac{g(y)}{|x-y|^n} dS(y)\\
u(x) &= \int_{\partial B(0,1)} K(x,y) g(y) dS(y)\end{aligned}$$

where,

$$K(x,y) = \frac{1-|x|^2}{n\alpha(n)} \frac{1}{|x-y|^2} \quad \text{(Poisson's Kernel).}$$

Now suppose u solve the boundary value problem

$$\triangle u = 0 \text{ in } B^0(0,r) \qquad (5)$$

$$u = g \text{ in } \partial B(0,r) \text{ for } r > 0$$

Then $\tilde{u}(x) = u(rx)$ with $\tilde{g}(x) = g(rx)$ solves $\triangle \tilde{u} = 0$ in $B(0,1)$ and $\tilde{u} = \tilde{g}$ on $\partial B(0,1)$. Now the formula for (5), becomes

$$u(x) = \frac{r^2 - |x|^2}{n\alpha(n)r} \int_{\partial B(0,r)} \frac{g(y)}{|x-y|^n} dS(y),\ x \in B^0(0,r)$$

Thus function

$$K(x,y) = \frac{r^2 - |x|^2}{n\alpha(n)r} \frac{1}{|x-y|^n}, \quad x \in B^0(0,r),\ y \in \partial B(0,r)$$

is Poisson's Kernel for the ball $B(0,r)$.

Theorem: Assume that $g \in C(\partial B(0,r))$ and u is defined by

$$u(x) = \frac{r^2 - |x|^2}{n\alpha(n)r} \int_{\partial B(0,r)} \frac{g(y)}{|x-y|^n} dS(y), \ x \in B^0(0,r),$$

then

$$(i) \ u \in C^\infty(B^0(0,r))$$
$$(ii) \ \triangle u = 0 \ \text{ in } \ B^0(0,r)$$
$$(iii) \ \lim_{x \longrightarrow x^0, x \in B^0(0,r)} = g(x^0) \ \text{ for each point } \ x^0 \in \partial B(0,r),$$

Question 6: Use Poisson's formula for the ball to prove

$$r^{n-2}\frac{r-|x|}{(r+|x|)^{n-1}}u(0) \leq u(x) \leq r^{n-2}\frac{r+|x|}{(r-|x|)^{n-1}}u(0)$$

whenever u is positive and harmonic in $B^0(0,r)$. This is an explicit form of Harnack's inequality.

Solution: when, $|x| < r$ and $|y| = r$, we get $r - |x| \leq |y - x| \leq r + |x|$ (by triangular inequality).

Poisson's formula gives us (and we must rely on $g \geq 0$) together with the above inequality plus the identity $r^2 - |x|^2 = (r - |x|)(r + |x|)$:

$$\begin{aligned} u(x) &= \frac{r^2 - |x|^2}{n\alpha(n)r} \int_{\partial B(0,r)} \frac{g(y)}{|x-y|^n} dS(y) \leq \frac{r^2 - |x|^2}{n\alpha(n)r} \int_{\partial B(0,r)} \frac{g(y)}{(r-|x|)^n} dS(y) \\ &= \frac{r+|x|)n\alpha(n)r^{n-1}}{n\alpha(n)r(r-|x|)^{n-1}} \int_{\partial B(0,r)} g(y)dS(y) = r^{n-2}\frac{r+|x|}{(r-|x|)^{n-1}}u(0), \end{aligned}$$

which is what we wanted. The other inequality is shown similarly, using the other half of the inequality of the first paragraph.

Energy Method:

Uniqueness of solution:

$$\left.\begin{aligned} -\triangle u = f \ \text{in} \ U \\ u = g \ \text{ in } \partial U \end{aligned}\right\} \qquad (1)$$

Let u_1 and u_2 solves the problem (1) and take

$$v = u_1 - u_2$$

Now v satisfies the following equation

$$\left.\begin{aligned} -\triangle v = 0 \ \text{in} \ U \\ v = 0 \ \text{ on } \partial U \end{aligned}\right\} \qquad (1)$$

Now, let us multiply this equation by v and integrating over U.

$$\int_U v \triangle v dx = 0$$

Now applying the following Green's formula

$$\int_{\partial U} f(x)(v(x).\nu)dS(x) = \int_U div(v(x))f(x)dx + \int_U gradf(x).v(x)dx$$

Take $f(x) = v, \quad v(x),$ we get

$$\int_{\partial U} v(Dv.\nu)dS(x) = \int_U div(Dv)vdx + \int_U grad(v).Dvdx$$

Now

$$\int_U v \triangle vdx = \int_{\partial U} v\frac{\partial v}{\partial \nu}dS(x) - \int_U (Dv).Dvdx$$

$$\int_U v \triangle vdx = \int_{\partial U} v\frac{\partial v}{\partial \nu}dS(x) - \int_U |Dv|^2dx$$

$$\because \int_U v \triangle vdx = 0$$

$$\implies \int_{\partial U} v\frac{\partial v}{\partial \nu}dS(x) - \int_U |Dv|^2dx = 0$$

As $v = 0$ on ∂U, we conclude that

$$\int_U |Dv|^2dx \quad = \quad 0,$$

Now $v = 0$ on ∂U and $Dv \equiv$ in U, implies that $v = 0$ in U, i.e., $u_1 = u_2$ proving the uniqueness of solution of (1).

Energy functional: Let us define energy functional as

$$I[w] = \int_U \left(\frac{1}{2}|Dw|^2 - wf\right) dx$$

and the class of Admissible function as

$$A = \{w \in C^2(\bar{U}); w = g \text{ on } \partial U\}$$

Theorem: A function $u \in C^2(\bar{U})$ solves the boundary value problem

$$\left.\begin{array}{r} -\triangle u = f \text{ in } U \\ \text{and } u = g \text{ on } \partial U \end{array}\right\} \qquad (1)$$

iff, $u \in A$ and $I[u] = \min_{w\in A} I[w]$

Proof: We have

$$\left.\begin{array}{r} -\triangle u = f \text{ in } U \\ \text{and } u = g \text{ on } \partial U \end{array}\right\} \qquad (1)$$

Let u solves (1), now take $w \in A$ and multiplying (1) by $u - w$ and integrating, we get

$$\int_U (-\triangle u - f)(u - w)dx = 0 \qquad (2).$$

Now applying the following Green's formula

$$\begin{aligned} \int_{\partial U} ((v(x)).\nu) f(x) dS(x) &= \int_U div(v(x)) f(x) dx + \int_U Df(x).v(x) dx \\ \text{by taking, } v(x) = Du, \quad f(x) &= u - w \end{aligned}$$

we get,

$$\int_{\partial U} (Du.\nu)(u - w) dS(x) = \int_U div(Du)(u - w) dx + \int_U D(u - w).Du dx$$

$$\text{or} \quad \int_U -\triangle u(x)(u - w) dx = \int_U Du.D(u - w) dx - \int_{\partial U} (Du.\nu)(u - w) dS(x),$$

since $u - w = 0$ on ∂U, therefore (2) becomes

$$\int_U [Du.D(u - w) - f(u - w)] dx = 0$$

$$\text{or} \quad \int_U [|Du|^2 - fu] dx = \int_U [Du.Dw - fw] dx.$$

By Cauchy-Schwarz inequality, we know that

$$|Du.Dw| \leq \frac{1}{2}|Du|^2 + \frac{1}{2}|Dw|^2$$

$$\int_U (|Du|^2 - fu) dx \leq \frac{1}{2} \int_U |Du|^2 dx + \int_U (\frac{1}{2}|Dw|^2 - fw) dx$$

or

$$\int_U (\frac{1}{2}|Du|^2 - fu) dx \leq \int_U \left(\frac{1}{2}|Dw|^2 - fw \right) dx$$

$$\implies I[u] \leq I[w]$$

$$\implies I[u] = \min_{w \in A} I[w]$$

Conversely, let $u \in A$ & $I[u] = \min_{w \in A} I[w]$, i.e. u be the minimizer of $I[w]$ over A. Take a function v that is smooth in U and vanishes on boundary ∂U. Consider the increment of $I[u]$ in the direction of v.

$$r(s) = I[u + sv] \qquad s \in \mathbb{R}$$

Then the function $u + sv$ is in A. As u minimizes $I[w]$ over A, we should have

$$r(s) \geq r(0) \quad \text{for all} \quad s \in \mathbb{R}$$

where
$$r(s)=\int_U\left\{\frac{1}{2}|Du+sDv|^2-(u+sv)f\right\}dx.$$

$$\begin{aligned}\because\ |Du+sDv|^2 &= (Du+sDv).(Du+sDv)\\ &= Du.Du+sDu.Dv+sDv.Du+s^2Dv.Dv\\ &= |Du|^2+2sDu.Dv+s^2|Dv|^2\\ r(s) &= \int_U\left(\frac{1}{2}|Du|^2-uf\right)dx+s\int_U(Du.Dv-vf)dx+\frac{s^2}{2}\int_U|Dv|^2dx\end{aligned}$$

Since function $r(s)$ is quadratic in s and attains its minimum at $s=0$ and so we have

$$\int_U(Du.Dv-vf)dx=0\ ??\ \ \text{(at minimum first derivative is zero)}$$

Integrating by parts and using that $v=0$ on ∂U gives

$$\int_U(-\triangle u-f)vdx=0$$

Since this identity holds for all smooth function v that vanishes at the boundary ∂U. It follows that u satisfies

$$-\triangle u=f \text{ in } U.$$

$$\text{Since } u\in A=\{w\in C^2(\bar{U});\ w=g \text{ on } \partial U\}$$

$$\text{this implies that } u=g \text{ on } \partial U$$

Hence

$$\left.\begin{array}{r}-\triangle u=f \text{ in } U\\ \text{and } u=g \text{ on } \partial U\end{array}\right\}$$

Heat equation: We study heat equation

$$u_t-\triangle u=0 \qquad (1)$$

and the non-homogeneous heat equation

$$u_t-\triangle u=f \qquad (2)$$

Subject to appropriate initial and boundary conditions. Here $t>0$ and $x\in u$ where $u\subset\mathbb{R}$ open. Then unknown is

$$u:\tilde{u}\times[0\ \infty)\longrightarrow\mathbb{R},\quad u=u(x,t)$$

and the Laplacian $\triangle$ is taken with respect to the spatial variables $x=(x_1,x_2,\ldots,x_n)$

$$\triangle_x u=\sum_{i=1}^n u_{x_ix_i},\ \text{the function } f:u\times[0\infty)\longrightarrow\mathbb{R}$$

is given.
Physical interpretation: The heat equation also known as the diffusion equation describes in typical applications the evaluation in time of the density u of some quantity such as heat, chemical concentration, etc.
If $V \subset U$ is any smooth subregion the rate of change of total quantity within V equals the negative of the net flux through ∂V.

$$\frac{d}{dt}\int_V u dx = -\int_{\partial V} \overrightarrow{F}.\nu ds$$

F being the flux density. Thus

$$u_t = -divF \qquad (3)$$

as V was arbitrary. In many situation F is proportional to the gradient of u. But points in the opposite direction (since the flow is from region of higher to lower concentration)

$$\overrightarrow{F} = -aDu \quad (a > 0)$$

$$\text{Now } u_t = adiv(Du) = a \triangle u$$

which for $a = 1$, is the heat equation.
The heat equation appears as well in the steady of Brownian motion.

Fundamental solution:
(a) Derivation of the fundamental solution:
We observe that the heat equation involves one derivative with respect to the time variable t, but two derivative with respect to the space variables $x_i (i = 1, 2, 3, \ . \ . \ . \ , n)$. Consequently, we see that if u solves (1) i.e. $u_t - \triangle u = 0 \quad (1)$
then so $u(\lambda x, \lambda^2 t)$ for $\lambda \in \mathbb{R}$
This scaling indices the ratio $\frac{r^2}{t}(r = |x|)$ is important for the heat equation and suggests that we search for a solution of (1) having the form

$$u(x,t) = v\left(\frac{r^2}{t}\right) = v\left(\frac{|x|^2}{t}\right) \quad (t > 0, x \in \mathbb{R})$$

It is quicker to seek a solution u having the special structure

$$u(x,t) = \frac{1}{t^\alpha} v\left(\frac{x}{t^\beta}\right) \;\; x \in \mathbb{R}, \;\; t > 0 \qquad (4)$$

where, $\alpha \& \beta$ are constant and the function $v : \mathbb{R} \longrightarrow \mathbb{R}$ must be found, we come to (4) if we look for a solution u of the heat equation invarient under dilation scaling.

$$u(x,t) \longrightarrow \lambda^\alpha u(\lambda^\beta x, \lambda t)$$

Now lwt us take

$$u(x,t) \longrightarrow \lambda^\alpha u(\lambda^\beta x, \lambda t)$$

for all

$$\lambda > 0, x \in \mathbb{R}, t > 0 \;\; \text{setting } \lambda = t^{-1}$$

$$\text{then } v(y) = u(y,1) \text{ for } y = t^{-\beta}x$$

$$\because u(x,t) = t^{-\alpha}v(xt^{-\beta})$$

By (1)

$$u_t - \triangle u = 0$$

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0$$

$$-\alpha t^{-\alpha-1}v(xt^{-\beta}) + t^{-\alpha}(-\beta t^{-\beta-1}x).Dv(xt^{-\beta}) - (t^{-\alpha} \triangle v(xt^{-\beta})t^{-2\beta}) = 0$$

$$\implies \alpha t^{-\alpha+1}v(y) + \beta t^{-(\alpha+1)}y.Dv(y) + t^{-(\alpha+2\beta)} \triangle v(y) = 0 \quad (5)$$

In order to transform (5) into an extression involving variables alone, we take $\beta = \frac{1}{2}$ then term with t are identical.

$$\alpha v + \frac{1}{2}y.Dv + \triangle v = 0 \qquad (6)$$

Now v to be a radial solution

$$\begin{aligned}
v(y) &= w(|y|) \\
v(y) &= w(r) \\
Dv(y) &= w'\frac{d}{dx_i}(r) \\
&= w'\frac{x_i}{r\sqrt{t}} \\
\text{Now,} \qquad \frac{1}{2}y.Dv(y) &= \frac{1}{2}y_i.w'\frac{x_i}{r\sqrt{t}} \\
&= \frac{w'}{r}\frac{1}{2}y_i.y_i = \frac{1}{2}\frac{w'}{r}|y|^2 \\
&= \frac{1}{2}rw'
\end{aligned}$$

$$\because v(y) = w(r), \ |y| = r = \frac{1}{\sqrt{t}}\sqrt{x_1^2 + \ . \ . \ . \ + x_n^2} = \sqrt{y_1^2 + \ . \ . \ . \ + y_n^2}$$

$$\frac{\partial v(y)}{\partial y_i} = w'(r)\frac{\partial r}{\partial x_i} = w'(r)\frac{x_i}{r}$$

$$\begin{aligned}
\frac{\partial^2 v}{\partial y_i^2} &= \left(\frac{w'(r)}{r} + y_i \frac{\partial}{\partial y_i}\left(\frac{w'(r)}{r}\right)\right) \\
&= \left(\frac{w'(r)}{r} + x_i \frac{\partial}{\partial r}\left(\frac{w'(r)}{r}\right)\frac{\partial r}{\partial y_i}\right) \\
&= \left(\frac{w'(r)}{r} + y_i \left(\frac{rw''(r) - w'(r)}{r^2}\right)\frac{x_i}{r}\right)\frac{1}{\sqrt{t}} \\
\triangle v = \sum_{i=1}^{n} v_{x_i x_i} &= \sum_{i=1}^{n} \frac{w'(r)}{r} + \frac{1}{r}\sum \frac{y_i^2}{t}\left(\frac{w'(r)}{r} - \frac{w'(r)}{r^2}\right) \\
&= \frac{nw'}{r} + \frac{|y|^2}{r}\left(\frac{w'(r)}{r} - \frac{w'(r)}{r^2}\right) \\
&= \frac{nw'}{r} + \left(\frac{rw''(r) - w'(r)}{r}\right) \\
&= \frac{(n-1)w'}{r} + w''(r)
\end{aligned}$$

Hence

$$\alpha w + \frac{1}{2} r w' + w'' + \frac{(n-1)}{r} w' = 0$$

for $r = |y|$. Now we set $\alpha = \frac{n}{2}$

$$\begin{aligned}
(r^{n-1}w')' + \frac{1}{2}(r^n w)' &= 0 \\
\alpha w + \frac{1}{2} r w' + w'' + \frac{(n-1)}{r} w' &= 0 \\
\frac{n}{2} w + \frac{1}{2} r w' + w'' + \frac{(n-1)}{r} w' &= 0 \\
\frac{n}{2} r^{n-1} w + \frac{1}{2} r^n w' + r^{n-1} w'' + \frac{(n-1)r^{n-1}}{r} w' &= 0 \\
\frac{1}{2}(nr^{n-1}w + r^n w') + (r^{n-1}w'' + (n-1)r^{n-2}w' &= 0 \\
\frac{1}{2}(r^n w)' + (r^{n-1}w')' &= 0
\end{aligned}$$

Integrating we get

$$\frac{1}{2}(r^n w) + (r^{n-1}w')' = a$$

where a is constant of integration

$$\lim_{r \longrightarrow \infty} w, w' = 0 \quad \text{so} \quad a = 0$$

Hence

$$\frac{1}{2} r^n w + r^{n-1} w' = 0$$

$$\Longrightarrow w' = -\frac{1}{2} r w$$

$$\Longrightarrow \frac{w'}{w} = -\frac{1}{2} r \Longrightarrow \log w = -\frac{r^2}{4} + \log b$$

$$\frac{w}{b} = e^{-\frac{r^2}{4}}$$

$$w = be^{-\frac{r^2}{4}} \qquad (7)$$

from (4) and (7)

$$u(x,t) = \frac{b}{t^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}$$

solves heat equation.

Definition: The function

$$\begin{cases} \frac{1}{(\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}} & x \in \mathbb{R}^n \quad t > 0 \\ 0 \quad u = g & x \in \mathbb{R}^n \quad t < 0 \end{cases}$$

is called the fundamental solution of heat equation. Notice that ϕ is singular at point $(0,0)$, we will some time write $\phi(x,t) = \phi(|x|,t)$ to emphasize that the fundamental solution is radial in the variables x.

<u>**Lemma:**</u> (Integral of fundamental solution) for each time $t > 0$

$$\int_{\mathbb{R}^n} \phi(x,t)dx = 1.$$

Proof: We calculate

$$\begin{aligned}
\int_{\mathbb{R}^n} \phi(x,t)dx &= \frac{1}{4\pi t}^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4t}} dx \\
&= \frac{1}{4\pi t}^{\frac{n}{2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \ . \ . \ . \ e^{-\frac{(x_1^2 + x_2^2 + \ . \ . \ . \ + x_n^2)}{4t}} dx_1 dx_2 \ . \ . \ . \ dx_n \\
&= \frac{1}{4\pi t}^{\frac{n}{2}} \left(\int_{-\infty}^{\infty} e^{-\frac{(x_1^2)}{4t}} dx_1 \right) \left(\int_{-\infty}^{\infty} e^{-\frac{(x_2^2)}{4t}} dx_2 \right) \ . \ . \ . \ \left(\int_{-\infty}^{\infty} e^{-\frac{(x_n^2)}{4t}} dx_n \right) \\
&= \frac{1}{4\pi t}^{\frac{n}{2}} \sqrt{4\pi t}\sqrt{4\pi t} \ . \ . \ . \ \sqrt{4\pi t}(n\text{times}) \\
&= \frac{1}{4\pi t}^{\frac{n}{2}} (4\pi t)^{\frac{n}{2}} = 1 \\
\int_{\mathbb{R}^n} \phi(x,t)dx &= 1
\end{aligned}$$

<u>**Initial Value Problem:**</u> A solution of the initial or cauchy problem

$$\begin{cases} u_t - \triangle u = 0 \text{ in } \mathbb{R} \times (0,\infty) \\ \quad u = g \text{ in } \mathbb{R} \times \{t = 0\} \end{cases}$$

Let us note the function $(x,t) \longrightarrow \phi(x,t)$ solves the heat equation away from the singularity at $(0,0)$ and thus so does $(x,t) \longrightarrow \phi(x-y,t)$ for each fixed $y \in \mathbb{R}$.

$$\begin{aligned}
u(x,t) &= \int_{\mathbb{R}} \phi(x-y,t)g(y)dy \\
&= \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}} \exp -\frac{|x-y|^2}{4t} g(y)dy \qquad (x \in \mathbb{R}, t > 0)
\end{aligned}$$

should also a solution.

Wave equation: The wave equation

$$u_{tt} - \triangle u = 0 \qquad (1)$$

and the non-homogeneous wave equation

$$u_{tt} - \triangle u = f \qquad (2)$$

subject to appropriate initial and boundary conditions. Here $t > 0$ and $x \in U$ where $U \subseteq \mathbb{R}$ is open. Then unknown is $U : \tilde{U} \times [0, \infty) \longrightarrow \mathbb{R},\ u = u(x,t)$ and Laplacian $\triangle$ is taken with respect to the spatial variables $x = (x_1, x_2, x_3, \ldots, x_n)$

Physical interpretation: The wave equation is a simplified model for a vibrating string $(n = 1)$ membrane $(n = 2)$ or elastic solid $(n = 3)$.

In these physical interpretation $u(x,t)$ represents the displacement in some direction of the point x at time $t \geq 0$.
Let V represent any smooth subregion of U. The acceleration within V is then

$$\frac{d^2}{dt^2}\int_V u dx = \int_V u_{tt} dx$$

and net contact force

$$-\int_{\partial V} \overrightarrow{F}.\nu ds$$

where $\overrightarrow{F}$ denotes force acting on V through ∂V and the mass density is taken to be unity. Newton's law asserts the mass times the acceleration equal to net force

$$\int_V u_{tt} dx = -\int_{\partial V} \overrightarrow{F}.\nu ds.$$

The identity obtained for each subregion V and so

$$u_{tt} = -div \overrightarrow{F}$$

For elastic bodies, $\overrightarrow{F}$ is a function of the displacement gradient Du;

$$u_{tt} + divF(Du) = 0, \quad \because F(Du) \simeq -\phi Du$$

$$u_{tt} - \phi \triangle u = 0.$$

This is the wave equation if $\phi = 1$

Solution by spherical means:

(a) Solution for $n = 1$, D'alembert's formula:

The initial-value problem for the one-dimensional wave equation in all of $\mathbb{R}$

$$\begin{cases} u_{tt} - u_{xx} = 0 \text{ in } \mathbb{R} \times (0, \infty) \qquad (3) \\ u = g, \ u_t = h \ \text{ on } \mathbb{R} \times \{t = 0\} \end{cases}$$

where g, h are given, we desire to derive formula for u in terms of g and h. Let us first note the P.D.E. (3) can be factored

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) u = u_{tt} - u_{xx} = 0 \qquad (4)$$

write

$$v(x,t) = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) u(x,t)$$

$$\Longrightarrow v_t(x,t) + v_x(x,t) = 0 \qquad (x \in \mathbb{R}, t > 0)$$

This is a non-homogeneous transport equation with $n = 1, b$